VERTICAL GENERA

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1. Introduction

In this paper we extend the classical constructions of genera on a compact boundaryless manifold, such as the signature, $\hat{A}$-genus and Todd genus, to the case of families of manifolds parametrised by a space X.

To do so requires the construction of certain generalised Pontryagin and Chern classes as maps from bordism cohomology to the singular cohomology of X. These relate to the usual characteristic classes on K-theory via the Chern-Dold character.

Bordism theory and genera are fundamental to both algebraic topology and geometric index theory, with broad implications for theoretical physics. Specifically, Classical (elliptic) genera define partition functions in type II superstring theory, while the classical bordism ring defines in its 'quantised' form the ontology of topological QFT. The parametrised genera considered here may be relevant to 'families' versions of these applications; for instance, it is natural to contemplate a fibred/vertical TQFT as a symmetric monoidal functor from the bordism category of fibre bundles to the category of vector bundles.

We begin by briefly reviewing the basic constructions of classical bordism.

1.1. **Classical genera.** A rational genus with values in an integral domain R over $\mathbb{Q}$ is a ring homomorphism

$$\varphi : \mathrm{MSO}_* \otimes \mathbb{Q} \to R \tag{1.1}$$

with MSO_* the oriented bordism ring. An element of MSO_* is the equivalence class $[N]$ of a closed oriented n-manifold N in which $N \sim N'$ (bordant) if $N \sqcup -N' = \partial W$ is the boundary of a compact $(n+1)$-manifold W, with $-N'$ the manifold N' with orientation reversed. The additive and multiplicative structure of MSO_* is defined by disjoint union and direct product, respectively. Thus, φ assigns to each oriented closed manifold N a element $\varphi(N) \in R \otimes \mathbb{Q}$ with $\varphi(N \sqcup N') = \varphi(N) + \varphi(N')$ and $\varphi(N \times N') = \varphi(N)\varphi(N')$, and such that if $N = \partial W$ is a boundary then $\varphi(N) = 0$. These properties are captured and characterised in a more granular way using the Pontryagin numbers

$$p_J(N) = \langle p_{j_1} \cdots p_{j_r}, [N] \rangle = \int_N p_{j_1} \cdots p_{j_r} \in \mathbb{Q} \tag{1.2}$$

defined for each partition $J = (j_1, \ldots, j_r)$ of $\frac{1}{4}\dim N \in \mathbb{N}$, with $p_k \in H^{4k}(N, \mathbb{Q})$ the kth Pontryagin class of N, along with a formal power series

$$f_\varphi \in R \otimes \mathbb{Q}[[x]] \tag{1.3}$$

prescribing how to construct the genus φ from linear combinations of the p_J. The intrinsic numbers $p_J(N)$ are cobordism invariants, in fact

$$[N] = [N'] \quad \text{in MSO}_* \otimes \mathbb{Q} \quad \Longleftrightarrow \quad p_J(N) = p_J(N') \;\; \forall\, |J| = \dim N/4, \tag{1.4}$$

equivalently, $p_{j_1}(N) \cup \cdots \cup p_{j_r}(N)$ with $j_1 \leq j_2 \leq \ldots \leq j_r$ and $\sum j_r = \dim N/4$ are a basis for the dual cobordism ring MSO^*. The characteristic power series f_ϕ has the form

$$f_\varphi(x) = \frac{x}{e_\varphi(x)} \in R \otimes \mathbb{Q}[[x]] \tag{1.5}$$

in which $e_\varphi \in R \otimes \mathbb{Q}[[x]]$, with leading term x and $e_\varphi(-x) = -e_\varphi(x)$, is the formal inverse power series to the logarithm series associated to φ defined by

$$l_\varphi(x) := \sum_{n=1}^{\infty} \frac{\varphi(\mathbb{C}P^{2n})}{2n+1}\, x^{2n+1} \;\; \in R \otimes \mathbb{Q}[[x]].$$

To give l_φ, or e_φ or f_φ, is the same thing as φ, as follows from Thom's identification $\text{MSO}_* \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \ldots]$, i.e. the spaces $\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \ldots$ form a generating set for $\text{MSO}_* \otimes \mathbb{Q}$; so, for example, $\text{MSO}_2 \otimes \mathbb{Q}$ is generated by $\mathbb{C}P^2$ while $\text{MSO}_4 \otimes \mathbb{Q}$ is generated by $\mathbb{C}P^2 \times \mathbb{C}P^2$ and $\mathbb{C}P^4$. Explicitly, the product $\Pi_{i=1}^N f_\varphi(x_i)$ is symmetric and even in each variable x_i and so each summand of homogeneity degree k can be written as a polynomial in the elementary symmetric functions of $x_1^2, \ldots, x_N^2$. On taking for x_i the Chern class of the ith splitting line bundle of the complexified tangent bundle, those elementary symmetric functions become the Pontryagin classes $p_k = p_k(TN)$. The component in $H^{\dim N}(N, \mathbb{Q})$ of the product gives in this way a polynomial $K(p_1, \ldots, p_k)$ and by a theorem of Hirzebruch

$$\varphi(N) = \langle K(p_1, \ldots, p_k), [N] \rangle.$$

The most familiar examples are the signature of a $n = 4k$ dimensional manifold for which $e_\varphi(x) = \tanh(x)$, and the $\hat{A}$-genus for which $e_\varphi(x) = 2\sinh(x/2)$. More recent work has been focused around the Witten genus $\omega : \text{MSO}_* \otimes \mathbb{Q} \to \mathbb{Q}[[q]]$ for which $e_\varphi(x) = 2\sinh(x/2)\Pi_{n\geq 1}(1 - q^n e^x)(1 - q^n e^{-x})(1 + q^n)^{-2}$, and its applications to modular forms, M-theory, and elliptic cohomology.

Such genera are natural evolutions of classical Riemann-Roch type formulae of algebraic geometry. This is made precise by geometric and topological index theory and the Atiyah-Singer index theorems. In the above cases, the signature is the index of the operator $d + d^* : \Omega^+(N) \to \Omega^-(N)$ on the even component of the Hodge decomposition of the de Rham complex, while if N is a spin manifold the $\hat{A}$ genus is equal to the index of the Dirac operator $\partial\!\!\!/ : \mathcal{S}^+(N) \to \mathcal{S}^-(N)$ on positive spinors. The Witten genus is formally the index of a Dirac operator on the loop space of a $4k$ dimensional spin manifold; a rigorous construction of this operator is missing, but there is a well developed stable homotopy description of the Witten genus as map of spectra from elliptic cohomology to topological modular forms [Hopkins].

1.2. **Vertical genera.** The most geometrically appealing case of the scenario we consider here is for a fibre bundle defined by a submersion $\pi : M \to X$ endowed with a choice of orientation on the vertical tangent bundle T_π over M (along the fibres), specifying a smooth family of closed oriented-diffeomorphic manifolds $N_x = \pi^{-1}(x)$, note M need not be orientable. A second such bundle $\pi' : M' \to X$ is fibrewise bordant to π if there is a vertically oriented (see below) manifold W with boundary diffeomorphic to $M \sqcup -M'$ and a map $\sigma : W \to X$ (we do not require σ to be a submersion) which restricts to π on M and π' on M'. The set of all such objects forms a graded ring $\eta\mathrm{SO}_\mathrm{X} = \bigoplus_q \eta\mathrm{SO}_{\mathrm{X,q}}$ summed over fibre dimension q in which the additive and multiplicative structure is defined by fibrewise disjoint union and fibre product. As a natural extension of (1.1), a fibred (vertical) rational genus is defined to be a ring homomorphism

$$\eta\mathrm{SO}_\mathrm{X} \otimes \mathbb{Q} \to H^*(X, \mathbb{Q}). \tag{1.6}$$

$\eta\mathrm{SO}_\mathrm{X}$ reduces when $X = pt$ is a point to the classical bordism ring MSO_* while (1.6) with $R = \mathbb{Q}$ then reduces to (1.1). An indication of the likely form of such genera is inferred from the cohomological version of the Atiyah-Singer index Theorem for a smooth family $\mathcal{D}_\pi^{\mathrm{spin}}$ of Dirac operators defined by a geometric spin fibration $\pi : M \to X$ (submersion) with index bundle $\mathrm{Ind}\,\mathcal{D}_\pi^{\mathrm{spin}} \in \mathrm{K}^0(X)$, which states that

$$\mathrm{ch}(\mathrm{Ind}\,\mathcal{D}_\pi^{\mathrm{spin}}) = \hat{A}(\pi) := \pi_!(\hat{A}(T_\pi)). \tag{1.7}$$

Here, $\pi_!(\hat{A}(T_\pi))$ – a vertical genus – is the the push-forward (integral over the fibre) to $H^*(X, \mathbb{Q})$ of the $\hat{A}$-genus in $H^*(M, \mathbb{Q})$ of the vertical tangent bundle T_π.

The construction of vertical genera does not, in fact, depend in any essential way on π being a submersion, a similar construction can be made for any homotopy class of maps $f : M \to X$ from a closed *vertically oriented* manifold M of dimension k to X, defining a bordism homology class $[f] \in \mathrm{MSO}_k(X)$ (see §2 for precise definitions). The natural inclusion

$$\eta\mathrm{SO}_{\mathrm{X,q}} \to \mathrm{MSO}_{\dim X + q}(X) \tag{1.8}$$

is then dual by Atiyah-Poincaré duality to an inclusion

$$\eta\mathrm{SO}_{\mathrm{X,q}} \to \mathrm{MSO}^{-q}(X) \tag{1.9}$$

identifying $\eta\mathrm{SO}_\mathrm{X}$ as a graded subring of bordism cohomology $\mathrm{MSO}^*(X)$. (The (co)bordism cohomology groups $\mathrm{MSO}^r(X)$ are defined for all integers r, non-trivially so for negative integers, but $\mathrm{MSO}^r(X) = 0$ for $r > \dim X$.) The upshot is that $\eta\mathrm{SO}_\mathrm{X}$ may be viewed either homologically or dually as a cohomology theory. For ring properties this is important since the graded multiplication defined by the fibre product

$$\eta\mathrm{SO}_{\mathrm{X,p}} \times \eta\mathrm{SO}_{\mathrm{X,q}} \to \eta\mathrm{SO}_{\mathrm{X,p+\,q}}$$

is seen to coincide via (1.9) with the cup product on $\mathrm{MSO}^*(X)$.

Consequently, the general definition of an oriented vertical genus over a smooth closed manifold X is a ring homomorphism

$$\nu_\mathrm{X} : \mathrm{MSO}^*(X) \otimes \mathbb{Q} \to \mathrm{H}^*(X, \mathbb{Q})$$

from the oriented cobordism cohomology ring $\mathrm{MSO}^*(X)$ to the singular cohomology ring of X. Our purpose here is to give a construction of such genera in terms of vertical Pontryagin classes

$$\mathsf{p}_J^{\mathrm{MSO}} : \mathrm{MSO}^*(X) \to \mathrm{H}^*(X, \mathbb{Z}), \qquad J = (j_1, \dots, j_m) \in \mathbb{N}^\infty,$$

which collectively characterise $\mathrm{MSO}^*(X) \otimes \mathbb{Q}$ in the same way that, classically (1.4), Pontryagin numbers characterise the classical oriented (co)bordism ring $\mathrm{MSO}_* \otimes \mathbb{Q} =$

$\mathrm{MSO}(pt)\otimes\mathbb{Q}$ coefficient ring to $\mathrm{MSO}^*(X)\otimes\mathbb{Q}$. Precisely, extending (1.4), one has:

Theorem 1. $\alpha=\alpha'$ in $\mathrm{MSO}^*(X)\otimes\mathbb{Q}\ \Leftrightarrow\ \mathsf{p}_J^{\text{MSO}}(\alpha)=\mathsf{p}_J^{\text{MSO}}(\alpha')$ in $\mathrm{H}^*(X,\mathbb{Q})\ \forall\, J\subset\mathbb{N}^\infty$.

A corresponding result for complex bordism cohomology $\mathrm{MU}^*(X)$ holds in terms of vertical Chern classes

$$\mathsf{c}_J^{\text{MU}}:\mathrm{MU}^*(X)\to\mathrm{H}^*(X,\mathbb{Z})$$

which relate to the Conner-Floyd Chern classes in $\mathrm{MU}^*(X)$ via the Chern-Dold character $\mathsf{ch}^{\text{MU}}:\mathrm{MU}^*(X)\to\mathrm{H}^*(X,\mathrm{MSO}_*)$. Vertical Chern classes can be built into stably complex vertical genera, defining ring homomorphisms $\mathrm{MU}^*(X)\otimes\mathbb{Q}\to\mathrm{H}^*(X,\mathbb{Q})$.

For unoriented bordism cohomology the correspondence holds in terms of vertical Stiefel-Whitney classes $\mathrm{sw}_J^{\text{MO}}:\mathrm{MO}^*(X)\to\mathrm{H}^*(X,\mathbb{Z}_2)$.

Vertical genera on $\mathrm{MSO}^*(X)\otimes\mathbb{Q}$, or in their geometrically natural habitat on $\eta\mathrm{SO}_{\mathrm{X}}$, admit a straightforward characterisation mirroring the classical case:

Theorem 2. *To each characteristic power series $f_\nu\in\mathbb{Q}[x]$, and resulting multiplicative sequence, there is a vertical genus $\nu_X:\mathrm{MSO}^*(X)\otimes\mathbb{Q}\to\mathrm{H}^*(X,\mathbb{Q})$ given by a power series in the vertical Pontryagin classes $\mathsf{p}_J^{\text{MSO}}$.*

An analogous statement holds for vertical stably complex genera. Each classical genus is thus seen to admit a vertical/fibrewise analogue. For example, one has the vertical $\hat{A}$-genus $\alpha\mapsto\hat{A}(\alpha)$ with component polynomials

$$\hat{A}_1(\alpha)=-\frac{1}{24}\mathsf{p}_1^{\text{MSO}}(\alpha),\ \ \hat{A}_2(\alpha)=\frac{7}{2^7\cdot3^2\cdot5}\left(-4\mathsf{p}_2^{\text{MSO}}(\alpha)+7\mathsf{p}_1^{\text{MSO}}(\alpha)^2\right),\dots\ \in\mathrm{H}^*(X,\mathbb{Q})$$

and so on. Theorem 2 says that familiy's index has the following multiplicativity property:

Corollary 1.1. *In $\mathrm{H}^*(X,\mathbb{Q})$ one has*

$$\hat{A}(\pi\times_X\pi')=\hat{A}(\pi)\cup\hat{A}(\pi') \tag{1.10}$$

for $\pi\times_X\pi':M\times_X M'\to X$ the fibre product (with fibre $\pi^{-1}(x)\times(\pi')^{-1}(x)$ at x).

Further, since rationally the Chern character ch is a ring isomorphism, the classical cobordism invariance of the pointwise index for a single manifold is repositioned for fibrations by Theorems 1 and 2 as:

Corollary 1.2. (Vertical cobordism invariance of the family index) *If $\pi:M\to X$ and $\pi':M'\to X$ are vertically bordant spin fibrations, so $[\pi]=[\pi']$, then*

$$\mathrm{Ind}\,\mathcal{D}_\pi^{spin}=\mathrm{Ind}\,\mathcal{D}_{\pi'}^{spin}\qquad in\ \mathrm{K(X)}\otimes\mathbb{Q}.$$

If $\pi:M\to X$ is a boundary fibration, then $\mathrm{Ind}\,\mathcal{D}_\pi^{spin}=0$. Moreover, for any π,π'

$$\mathrm{Ind}\,\mathcal{D}_{\pi\times_X\pi'}^{spin}=\mathrm{Ind}\,\mathcal{D}_\pi^{spin}\otimes\mathrm{Ind}\,\mathcal{D}_{\pi'}^{spin}.$$

Similarly, there is a vertical L-genus which from [Atiyah2] implies the vertical bordism invariance of a natural geometric family of signature operators $\mathcal{D}_\pi^{\text{sig}}$ and the multiplicativity of its index bundle. It would be interesting to know more about other vertical elliptic genera from the viewpoint here, such as the vertical Witten genus taking values in $H^*(X, \mathbb{Q}[[q]])$, which for a fibre bundle $M \xrightarrow{\pi} X$ of $4n$-dimensional closed spin manifolds may, conjecturally, be close to the Chern character of the formal index bundle of the resulting family of Witten-type Dirac operators on the fibrewise loop space of M.

2. Vertically oriented bordism homology

All manifolds will be smooth and compact. The tangent bundle of a manifold N will be denoted T_N. The orientation bundle $\mathcal{O}_N$ of N is the principal $\mathrm{Gl}(1,\mathbb{R})$ frame bundle $F(\mathrm{Det}\, T_N)$ of the determinant line bundle $\mathrm{Det}\, T_N := \wedge^n T_N$, or, equivalently for our purposes, the principal $\mathbb{Z}_2$ bundle $F(\mathrm{Det}\, T_N)_{\mathrm{Gl}(1,\mathbb{R})} \times \mathbb{Z}_2$ coming from the map $\mathrm{Gl}(1,\mathbb{R}) \to \mathbb{Z}_2$, $x \mapsto x/|x|$, whose transition functions are $\det(g_{ij})/|\det(g_{ij})|$ with g_{ij} the transition functions of T_N. More generally, the orientation bundle $\mathcal{O}_{E-F}$ of a stable bundle $E - F$ in the K-theory ring of real vector bundles $\mathrm{KO}(X)$ is the frame bundle of the determinant line bundle $\mathrm{Det}\, E \otimes \mathrm{Det}\, F^*$, and a choice of orientation on $E - F$, if one exists, is a choice of trivialisation of $\mathcal{O}_{E-F}$.

A smooth map $f : N \to X$ is said to be vertically oriented if there exists an isomorphism of orientation bundles $\mathcal{O}_N \cong f^*\mathcal{O}_X$ with a choice of isomorphism $\alpha_{\mathrm{N,X}} : \mathcal{O}_N \to f^*\mathcal{O}_X$ specified. One has:

Lemma 2.1. *A vertical orientation on $f : N \to X$ is equivalent to a choice of orientation on its stable normal bundle*

$$\mathcal{V}_f^{\mathrm{st}} := f^*T_X - T_N \ \in \mathrm{KO}(N).$$

Proof. $f^*\mathcal{O}_X = \mathcal{O}_{f^*T_X}$ and so a vertical orientation on f is equivalent to a choice of isomorphism $\mathrm{Det}\, T_N \cong f^*\mathrm{Det}\, T_X \cong \mathrm{Det}\, f^*T_X$, that is, a trivialisation of the line bundle $\mathrm{Det}\, T_N \otimes \mathrm{Det}\, f^*T_X$, which is the same as a trivialisation of its frame bundle. □

The induced orientation in a fibre product is defined as follows. Let $f : M \to X$ and $g : N \to X$ be smooth maps. Then there is a commutative diagram

(2.1)
$$\begin{tikzcd} M \times_X N \arrow[r, "\mu"] \arrow[d, "\nu"] \arrow[dr, "s = f \times_X g"] & M \arrow[d, "f"] \\ N \arrow[r, "g"] & X \end{tikzcd}$$

in which

$$M \times_X N = \{(y,z) \in M \times N \mid f(y) = g(z)\}$$

is the fibre product of (M, f) and (N, g). Equivalently, $M \times_X N = (f \times g)^{-1}(\mathrm{diag}(X))$ with $\mathrm{diag}(X)$ the diagonal in $X \times X$. The maps μ, ν are the projection maps. The diagonal map is, thus,

$$s = f \circ \mu = g \circ \nu. \tag{2.2}$$

f and g are transverse if for $(y,z) \in M \times_{\mathrm{X}} N$ one has $df_y(T_yM) + dg_z(T_zN) = T_xX$, where $x = f(y) = g(z)$, or, equivalently, if $f \times g$ is transverse to $\mathrm{diag}(X)$. If f and g are transverse maps then $M \times_X N$ is a smooth manifold of dimension $\dim M + \dim N - \dim X$; if $\dim X > \dim M + \dim N$ then $M \times_X N$ is the empty manifold (see Prop II.4 [Lang]).

Proposition 2.2. *Let f and g be transverse maps. Then*

$$\mathcal{V}^{\rm st}_{\nu} = \mu^*\mathcal{V}^{\rm st}_{f} \tag{2.3}$$

and

$$\mathcal{V}^{\rm st}_{f\times_X g} = \mu^*\mathcal{V}^{\rm st}_{f} + \nu^*\mathcal{V}^{\rm st}_{g}. \tag{2.4}$$

The fibre product map $f\times_X g : M\times_X N\to X$ is canonically vertically oriented by vertical orientations on $f : M\to X$ and $g : N\to X$. The pull-back map

$$\nu : g^*(M\xrightarrow{f} X) := M\times_X N\to N$$

is canonically vertically oriented by a vertical orientation on $f : M\to X$.

Proof. For transverse f and g there is a bundle isomorphism

$$T_{M\times_X N} + s^*T_X \cong \mu^*T_M + \nu^*T_N, \tag{2.5}$$

as follows from the exact sequence [Joyce]

$$0\to T_{M\times_X N}\xrightarrow{d\mu\oplus d\nu}\mu^*T_M+\nu^*T_N\xrightarrow{\mu^*df-\nu^*dg} s^*(T_X)\to 0.$$

Exactness is by (2.2) and transversality. Hence as stable bundles

$$T_{M\times_X N} - \nu^*T_N \overset{(2.5)}{=} \mu^*T_M - s^*T_X = \mu^*T_M - (f\circ\mu)^*T_X = \mu^*(T_M - f^*T_X),$$

which is (2.3), and

$$T_{M\times_X N} - s^*T_X = (\mu^*T_M - s^*T_X) + (\nu^*T_N - s^*T_X) \tag{2.6}$$

which from (2.2) is (2.4). By Lemma 2.1 the stated induced orientations are immediate from taking top exterior powers (determinant bundles) of these identifications. □

From Atiyah's construction [Atiyah1], an element $[f,N,\mu]\in \mathrm{MSO}_n(X)$ of bordism homology is represented by a vertically oriented map $f : N\to X$ from a closed manifold N of dimension n — thus with a given orientation bundle isomorphism $\mu : f^*\mathcal{O}_X\cong\mathcal{O}_N$ — with two such triples (f,N,μ), (f',N',μ'), being equivalent if there is a triple (σ,W,λ) with $\sigma : W\to X$ an oriented map from a manifold W of dimension $n+1$ with boundary $\partial W\cong N\sqcup N'$ such that $\sigma_{|N}=f$, $\sigma_{|N'}=f'$, and $\lambda : \sigma^*\mathcal{O}_X\cong\mathcal{O}_W$ with $\lambda_{|N}=\mu$, $\lambda_{|N'}=-\mu'$.

Comment 2.3. In fact, Atiyah defines a bordism homology group $\mathrm{MSO}_n(X,\alpha)$ for each principal $\mathbb{Z}_2$-bundle α on X. In that notation, $\mathrm{MSO}_n(X)$ is the group $\mathrm{MSO}_n(X,\mathcal{O}_X)$ — note, in contrast to its usage here, in [Atiyah1] '$\mathrm{MSO}_n(X)$' refers to the case $\mathcal{O}_X = X\times\mathbb{Z}_2$, i.e. orientable X.

The additive abelian structure of $\mathrm{MSO}_n(X)$ is defined by the fibrewise disjoint union $(f\sqcup_{\rm X} f', N\sqcup_{\rm X} N', \alpha\sqcup_{\rm X}\alpha')$. Fibre product defines a natural product

$$\mathrm{MSO}_m(X)\times\mathrm{MSO}_n(X)\to\mathrm{MSO}_{m+n-\dim \mathrm{X}}(X), \tag{2.7}$$

and for $\phi : Y\to X$ a Umkehr/Gysin/transfer map

$$\mathrm{MSO}_m(X)\to\mathrm{MSO}_{m+\mathrm{dimY}-\,\mathrm{dimX}}(Y). \tag{2.8}$$

The product (2.7) maps $([f,M,\mu],[f',M',\mu'])$ to $[f\times_{\rm X} f' : M\times_X M', \mu\times_{\rm X}\mu']$ where f, f' are chosen (homotoped) within their bordism classes to be mutually transverse. Similarly, (2.8) is the map $[f,M,\alpha]\mapsto[f\times_{\rm X}\phi, M\times_X Y, \phi^*\alpha]$ with f chosen transverse to ϕ. $\mu\times_X\mu'$ and $\phi^*\alpha$ indicate the canonical induced orientations of Proposition 2.2. See [Cohen, Klein] for more on Umkehr maps.

(2.7) is not a graded product [1]. This is corrected by Atiyah-Poincaré duality which posits matters into bordism cohomology, with respect to which the homology fibre product (2.7) relates to the product on the graded bordism cohomology ring in the same way that the intersection product $\mathrm{H}_m(X)\times\mathrm{H}_n(X)\to\mathrm{H}_{m+n-\dim X}(X)$ on singular homology relates to the cup product via classical Poincaré duality. Precisely, one has a commutative array

$$\begin{array}{ccc} \mathrm{MSO}^{-q}(X)\times\mathrm{MSO}^{-q'}(X) & \longrightarrow & \mathrm{MSO}^{-(q+q')}(X) \\ \uparrow & & \uparrow \\ \mathrm{MSO}_{\dim X+q}(X)\times\mathrm{MSO}_{\dim X+q'}(X) & \longrightarrow & \mathrm{MSO}_{\dim X+q+q'}(X) \end{array} \tag{2.9}$$

in which the duality map $\mathrm{MSO}_{\dim X+q}(X)\to\mathrm{MSO}^{-q}(X)$ is defined as follows. For vertically oriented $f:M\to X$ representing $[f,M,\alpha]\in\mathrm{MSO}_{\dim X+q}(X)$, with orientation α on its stable normal bundle

$$\mathcal{V}_f^{\mathrm{st}}=f^*T_X-T_M$$

of rank

$$\mathrm{rk}(\mathcal{V}_f^{\mathrm{st}})=-q:=-(\dim M-\dim X),$$

we may choose an embedding $e_l:M\to\mathbb{R}^l$ and fibrewise it to the embedding

$$e_l\times f:M\to\mathbb{R}^l\times X,\qquad m\mapsto(e(m),f(m)),$$

which extends to an embedding of the normal bundle

$$\mathcal{V}_{e_l\times f}=(e_l\times f)^*(\underline{l}+T_X)/T_M\to\mathbb{R}^l\times X \tag{2.10}$$

of $e_l\times f$, with $\underline{l}$ the rank l trivial bundle. $\mathcal{V}_{e_l\times f}$ has rank

$$\mathrm{rk}(\mathcal{V}(e_l\times f))=l-q.$$

The Thom construction collapses (2.10) contravariantly to the Pontryagin-Thom map

$$\Sigma^l X_+\to M^{\mathcal{V}_{e_l\times f}}. \tag{2.11}$$

Here, $Y^V=\mathrm{Th}(V)$ is the Thom space (one-point compactification of the total space) of a vector bundle $V\to Y$; in particular, the iterated reduced suspension $\Sigma^l X_+$ is the Thom space of the trivial bundle $\underline{l}=\mathbb{R}^l\times X$.

Since $\mathcal{V}_{e_l\times f}-\underline{l}$ is a representative for the stable normal bundle $\mathcal{V}_f^{\mathrm{st}}$ this can be positioned invariantly. $\mathcal{V}_{e_l\times f}$ is the same thing as the corresponding homotopy class of maps $M\to\mathrm{BSO}_{l-q}$ to the Grassmannian of oriented $l-q$ planes, covered by a map $\mathcal{V}_{e_l\times f}\to\xi_{l-q}$ to the universal bundle, giving an induced map of Thom pre-spectra $M^{\mathcal{V}_{e_l\times f}}\to\mathrm{MSO}(l-q)$, where $\mathrm{MSO}(n):=\mathrm{BSO}_n^{\xi_n}$. With (2.11) this defines an element of $[\Sigma^l X_+,\mathrm{MSO}(l-q)]$, which as a stable colimit is an element of

$$\mathrm{MSO}^{-q}(X):=\varinjlim_l[\Sigma^l X_+,\mathrm{MSO}(l-q)].$$

[1]Except on the subgroup $\eta so_{X,q}$ of fibre bundles of fibre dimension q for which (2.7) defines a graded product $\eta so_{X,q}\times\eta so_{X,q'}\to\eta so_{X,q+q'}$ coinciding with the cohomology product via (1.9).

If we work in the stable category we may desuspend the map (2.11) directly to get the stable Pontryagin-Thom map

$$X_+ \to \Sigma^{-l} M^{\mathcal{V}_{e_l\times f}} = M^{\mathcal{V}_{e_l\times f}-\underline{l}} = M^{\mathcal{V}_f^{\mathrm{st}}} \tag{2.12}$$

to the Thom space of the rank $-q$ stable bundle $\mathcal{V}_f^{\mathrm{st}}$ and so an element in

$$\mathrm{MSO}^{-q}(X) = [X_+, \mathrm{MSO}_{-q}],$$

where MSO_{-q} is the Thom spectrum (proper), i.e. the suspension spectrum of the stable bundle in $\mathrm{K(BSO)}$ of rank 0 which restricts on BSO_n to $\xi_n - \underline{n}$. As the graded ring product on $\mathrm{MSO}^*(X)$ is induced by Whitney sum, the commutativity of (2.9) follows from (2.4). The proof that the homotopy class so constructed is independent of the bordism class of f and that the assignment is bijective follows closely standard Thom constructions and the use of L-equivalence to mediate the duality map, similarly to [Atiyah1].

There is likewise a Pontryagin-Thom map between the Thom complexes of the stable normal bundles

$$X^{-T_X} \to M^{-T_M}, \tag{2.13}$$

which is a particular instance of the extension to the case of a virtual bundle $\xi \to X$, for which one has the Pontryagin-Thom map

$$X^{\xi} \to M^{f^*\xi + \mathcal{V}_f^{\mathrm{st}}}$$

obtained by applying (2.11) to the induced map $f^*\xi \to \xi$. Setting $\xi = \zeta - T_X$ for a virtual bundle ζ on X, this can be put into the form [Crabb, James], [Cohen, Jones],

$$X^{\zeta - T_X} \to M^{f^*\zeta - T_M}$$

which yields (2.12) and (2.13) efficiently.

A similar characterisation holds for complex bordism homology $MU_m(X)$ comprising bordism classes $[M, f]$ of maps $f : M \to X$, $\dim M = m$, with a stable complex structure on $\mathcal{V}_f^{\mathrm{st}} = f^*T_X - T_M$, meaning a class of complex structures on $f^*T_X + \mathcal{E}$ where $T_M + \mathcal{E} = \underline{n}$, i.e. $\mathcal{V}_f^{\mathrm{st}}$ is in the image of the forgetful map $\mathrm{K}(X) \to \mathrm{KO}(X)$. There is a natural inclusion $\eta\mathrm{U}_{\mathrm{X,q}} \to \mathrm{MU}_{q+\dim X}(X)$. On the other hand, with $q = \dim M - \dim X$, we may choose an embedding $M \to \mathbb{R}^l$ with an isomorphism $\mathcal{V}_{e_l\times f} = \zeta_{\mathbb{R}}$ with ζ a stable complex vector bundle of rank $l - q = 2k$ identified with the pullback $\mu^*(\xi_k)$ of the canonical bundle $\xi_k \to \mathrm{BU}(k)$ for a homotopy class $\mu \to \mathrm{BU}(k)$. Taking Thom spaces yields a map $\Sigma^l X_+ \to M^{\mathcal{V}_{e_l\times f}} \to \mathrm{MU}(k)$ and so the duality isomorphism $\mathrm{MU}_q(X) \to \mathrm{MU}^{\dim X - q}(X)$ to the stable homotopy group

$$\mathrm{MU}^n(X) = \varinjlim_k [\Sigma^{2k-n} X_+, \mathrm{MU}(k)].$$

3. Umkehr for vertically oriented maps

Let $\mathbb{E}$ be a multiplicative cohomology theory. An $\mathbb{E}$-orientation on a vector bundle $V \xrightarrow{\mu} M$ of rank n is a Thom class $u = u_V \in \tilde{\mathbb{E}}(M^V)$ whose fibrewise restriction

$$\tilde{\mathbb{E}}^n(M^V) \to \tilde{\mathbb{E}}^n(\mathrm{Th}(V_m)) = \tilde{\mathbb{E}}^n(S^n)$$

is a generator for each m in M. One then has a Thom isomorphism

$$\phi = \phi_V : \mathbb{E}^r(M) \to \tilde{\mathbb{E}}^{r+n}(M^V), \quad c \mapsto \mu^*(c) \cup u_V, \quad u_V = \phi_V(1). \tag{3.1}$$

This extends to a canonical isomorphism

$$\phi_{V-\underline{l}} : \mathbb{E}^r(M) \to \tilde{\mathbb{E}}^{r+n}(M^V) \cong \tilde{\mathbb{E}}^{r+n-l}(\Sigma^{-l}M^V) := \tilde{\mathbb{E}}^{r+n-l}(M^{V-\underline{l}})$$

with the first arrow implemented as in (3.1), and hence to defining a stable Thom isomorphism for virtual bundles via $M^{W-W'} = M^{W+W''-\underline{l}}$ where $W'+W'' = \underline{l}$ - note that n may be negative - this is used here only for the stable normal bundle.

A vertical $\mathbb{E}$-orientation on a map $f : M \to X$ is defined to be an $\mathbb{E}$-orientation on its stable normal bundle $\mathcal{V}_f^{\rm st}$, or, equivalently, compatibly on each normal bundle $\mathcal{V}_{e_l\times f}$. One then has a Thom isomorphism $\mathbb{E}^r(M) \to \tilde{\mathbb{E}}^{r+l-q}(M^{\mathcal{V}_{e_l\times f}})$ which combines with the Pontryagin-Thom map (2.11) to define the Umkehr/Gysin/integration-over-the-fibre map

$$f_! : \mathbb{E}^r(M) \to \tilde{\mathbb{E}}^{r+l-q}(M^{\mathcal{V}_{e_l\times f}}) \xrightarrow{\text{P.T.}} \tilde{\mathbb{E}}^{r+l-q}(\Sigma^l X_+) \xrightarrow{\cong} \mathbb{E}^{r-q}(X), \tag{3.2}$$

or, more cleanly, by combining with the stable Pontryagin-Thom map (2.12) as

$$f_! : \mathbb{E}^r(M) \to \tilde{\mathbb{E}}^{r-q}(M^{\mathcal{V}_f^{\rm st}}) \xrightarrow{\text{P.T.}} \mathbb{E}^{r-q}(X), \tag{3.3}$$

and in case M is itself $\mathbb{E}$-orientable using (2.13) as

$$f_! : \mathbb{E}^r(M) \to \tilde{\mathbb{E}}^{r-\dim M}(M^{-T_M}) \xrightarrow{\text{P.T.}} \tilde{\mathbb{E}}^{r-\dim M}(X^{-T_X}) \to \mathbb{E}^{r-\dim M+\dim X}(X), \tag{3.4}$$

where unmarked arrows in (3.2), (3.3), (3.4) are Thom isomorphisms, noting that T_M and $\mathcal{V}_f^{\rm st}$ are $\mathbb{E}$-orientable imply T_X is $\mathbb{E}$-orientable. The Umkehr map $f_!$ is functorial with

$$f_!(f^*(\sigma) \cup \omega) = \sigma \cup f_!(\omega). \tag{3.5}$$

In the case (3.4), when M and X are $\mathbb{E}$-orientable, $f_!$ may be constructed in its classical form as the Poincaré dual to the push-forward f_* in homology

$$f_! : \mathbb{E}^r(M) \xrightarrow{D} \mathbb{E}_{\dim M-r}(M) \xrightarrow{f_*} \mathbb{E}_{\dim M-r}(X) \xrightarrow{D^{-1}} \mathbb{E}^{\dim X-\dim M+r}(X). \tag{3.6}$$

The Poincaré duality isomorphisms D are defined by cap product with the respective $\mathbb{E}$ fundamental homology classes defined by the orientations on M and X. More generally, Poincaré duality in $\mathbb{E}$ holds provided that the manifold X is oriented in $\mathbb{E}$ and Spanier-Whitehead duality (S-duality) holds, then the diagram of isomorphisms

$$\begin{tikzcd} \mathbb{E}^r(X) \arrow[r, "\text{Thom}"] \arrow[dr, "D"'] & \tilde{\mathbb{E}}^{r-\dim X}(X^{-T_X}) \arrow[d, "S"] \\ & \mathbb{E}_{\dim X-r}(X) \end{tikzcd} \tag{3.7}$$

commutes. Lemma 2.1 expresses the equivalence in singular cohomology of the Thom-Pontryagin (3.3) and Poincaré duality (3.6) realisations of $f_!$.

Comment 3.1. Atiyah-Poincaré duality holds on the multiplicative cohomology MSO^* in Sect.1 without X being oriented, vertical orientability suffices, using just the Pontryagin-Thom map. In the presence of an orientation this coincides with (3.7) just as the Umkehr map (3.3), defined without an orientation on X, then coincides with (3.4). The isomorphism $\mathrm{MSO}_n(X) := \mathrm{MSO}_n(X, \mathcal{O}_X) \cong \mathrm{MSO}^{\dim X-n}(X)$ of

[Atiyah1] may be viewed as Poincaré duality for homology with twisted coefficients in a comparable way to classical singular homology Poincaré duality for non-oriented manifolds [Ranicki].

The fundamental property of the Umkehr for vertically oriented maps $f : M \to X$ is functoriality:

Proposition 3.2. *For $\phi : Y \to X$ let $\mu : \phi^*(M) = M \times_X Y \to Y$ be the fibre product pull-back, for which one has the commutative diagram (2.1) relabelled here as*

$$\begin{tikzcd} W := M \times_X Y \arrow[r, "\beta"] \arrow[d, "\mu"] & M \arrow[d, "f"] \\ Y \arrow[r, "\phi"] & X. \end{tikzcd} \tag{3.8}$$

Then,

$$\mu_! \circ \beta^* = \phi^* \circ f_!. \tag{3.9}$$

Proof. Frobenius reciprocity (Prop 12.9 [Crabb, James]) applied to (3.8) gives a commutative diagram of stable maps

$$\begin{tikzcd} Y_+ \arrow[r, "\phi_+"] \arrow[d, "\mu_+"] & X_+ \arrow[d, "f_+"] \\ W^{\mathcal{V}_\mu^{\rm st}} \arrow[r, "\beta_+"] & M^{\mathcal{V}_f^{\rm st}}, \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \mathbb{E}^{r-q}(M^{\mathcal{V}_f^{\rm st}}) \arrow[r, "f_+^*"] \arrow[d, "\beta_+^*"] & \mathbb{E}^{r-q}(X) \arrow[d, "\phi^*"] \\ \mathbb{E}^{r-q}(W^{\mathcal{V}_\mu^{\rm st}}) \arrow[r, "\mu_+^*"] & \mathbb{E}^{r-q}(Y), \end{tikzcd}$$

which concatenates with the naturality of the Thom isomorphism — that since $\mathcal{V}_\mu^{\rm st} = \beta^* \mathcal{V}_f^{\rm st}$ (Prop 2.2)

$$\begin{tikzcd} \mathbb{E}^r(M) \arrow[r, "\text{Thom}"] \arrow[d, "\beta^*"] & \mathbb{E}^{r-q}(M^{\mathcal{V}_f^{\rm st}}) \arrow[d, "\beta_+^*"] \\ \mathbb{E}^r(W) \arrow[r, "\text{Thom}"] & \mathbb{E}^{r-q}(W^{\mathcal{V}_\mu^{\rm st}}) \end{tikzcd}$$

commutes — to give the commutativity of

$$\begin{tikzcd} E^r(M) \arrow[r, "\text{Thom}"] \arrow[d, "\beta^*"] & \mathbb{E}^{r-q}(M^{\mathcal{V}_f^{\rm st}}) \arrow[r, "f_+^*"] \arrow[d, "\beta_+^*"] & \mathbb{E}^{r-q}(X) \arrow[d, "\phi^*"] \\ E^r(W) \arrow[r, "\text{Thom}"] & \mathbb{E}^{r-q}(W^{\mathcal{V}_\mu^{\rm st}}) \arrow[r, "\mu_+^*"] & \mathbb{E}^{r-q}(Y), \end{tikzcd}$$

and hence (3.9). □

4. Vertical characteristic classes

We may apply the Umkehr/Gysin/integration-over-the-fibre for vertically oriented maps to define for each multi-index $J = (j_1, j_2, \ldots, j_r) \subset \mathbb{N}^\infty$ a generalised characteristic class

$$\mathsf{p}_J^{\rm MSO} : \mathrm{MSO}^*(X) \otimes \mathbb{Q} \to \mathrm{H}^*(X, \mathbb{Q})$$

by

$$\mathsf{p}_J^{\text{MSO}}(\omega) = f_!^{\omega}(\mathsf{p}_J(-\mathcal{V}_{f^\omega}^{\text{st}})), \tag{4.1}$$

where $f^\omega : M \to X$ is a representative for the dual of $\omega = D^{\text{MSO}}[f^\omega]$, with D^{MSO} the Atiyah-Poincaré duality isomorphism of Sect.1 – this is independent of the choice of $f^\omega \in [f^\omega]$, see Theorem 4.3. $[f]$ abbreviates a homology class $[f, M, \alpha] \in \mathrm{MSO}_*(X) \otimes \mathbb{Q}$. On the right-hand side, $\mathsf{p}_J(\zeta) = p_{j_1}(\zeta) \cdots p_{j_r}(\zeta) \in \mathrm{H}^{4|J|}(M, \mathbb{Q})$ is the J^{th} Pontryagin class of a stable bundle $\zeta \in \mathrm{KO}(M)$.

Here, a characteristic class $\alpha(E) = 1 + \alpha_1(E) + \cdots + \alpha_m(E)$, $\alpha_r(E) \in \mathrm{H}^r(M, R)$, on a semigroup of vector bundles E which on trivial bundles is equal to 1 and has the exponential property $\alpha(E + F) = \alpha(E) \cdot \alpha(F)$ is stable provided there are no odd degree terms $\alpha_{2j+1}(E)$, or without restriction on the $\alpha_r(E)$ if $R = \mathbb{Z}_2$. In these cases α extends to virtual bundles $E - E'$ in the corresponding K-theory ring as $\alpha(E - E') := \alpha(E)/\alpha(E')$ in $\mathrm{H}^*(M, R)$, so $\alpha_0(E - E') = 1$, $\alpha_1(E - E') = \alpha_1(E) - \alpha_1(E')$, $\alpha_2(E - E') = \alpha_2(E) - \alpha_1(E)\alpha_1(E') - \alpha_2(E')$, and so on. The total Chern class, the rational total Pontryagin class, and the total Stiefel-Whitney classes are each stable classes and defined on the corresponding stable normal bundle $\mathcal{V}_f^{\text{st}}$.

The rational total Pontryagin class has the form

$$\mathsf{p}(-\mathcal{V}_f^{\text{st}}) := p(T_M - f^* T_X) = 1 + \mathsf{p}_1(-\mathcal{V}_f^{\text{st}}) + \mathsf{p}_2(-\mathcal{V}_f^{\text{st}}) + \cdots$$

with $\mathsf{p}_j(-\mathcal{V}_f^{\text{st}}) \in \mathrm{H}^{4j}(M, \mathbb{Z})$, and

$$\mathsf{p}_J(-\mathcal{V}_f^{\text{st}}) := \mathsf{p}_{j_1}(-\mathcal{V}_f^{\text{st}})\mathsf{p}_{j_2}(-\mathcal{V}_f^{\text{st}}) \cdots \mathsf{p}_{j_r}(-\mathcal{V}_f^{\text{st}}) \in \mathrm{H}^{4|J|}(X, \mathbb{Z}).$$

The Umkehr map of this is the vertical characteristic class (4.1). If $f = \pi$ is a submersion defining $[\pi] \in \eta\mathrm{SO}_{\mathrm{X}}$ then

$$\mathsf{p}_J^{\text{MSO}}(D^{\text{MSO}}[\pi]) := \pi_!(\mathsf{p}_J(T_\pi M)) = \int_{M/X} \mathsf{p}_J(M/X)$$

associated to the Pontryagin class of the tangent bundle along the fibres $T_\pi M$. The notation $\int_{M/X}$ refers to the de Rham realization of $\pi_!$ as integration over the fibre on a Chern-Weil representative $\mathsf{p}_J(M/X)$ of the given classes.

The choice of orientation α in $[f^\alpha] := [f] = [f, M, \alpha]$ has been suppressed in the above, but the vertical Pontryagin class is sensitive to it, even though $\mathsf{p}_J(-\mathcal{V}_f^{\text{st}})$ is not. When the orientation α on $\mathcal{V}_f^{\text{st}}$ is reversed to $-\alpha$ the Thom class then changes sign and hence

$$f_!^{-\alpha} = -f_!^{\alpha}\,,$$

and hence:

Lemma 4.1.

$$\mathsf{p}_J^{\text{MSO}}(D^{\text{MSO}}[f, M, -\alpha]) = -\,\mathsf{p}_J^{\text{MSO}}(D^{\text{MSO}}[f, M, \alpha]). \tag{4.2}$$

We note also the naturality of the vertical classes.

Proposition 4.2. *For a map $\phi : Y \to X$ one has for each $\omega \in \mathrm{MSO}^*(X)$*

$$\phi^* \mathsf{p}_J^{\text{MSO}}(\omega) = \mathsf{p}_J^{\text{MSO}}(\phi^* \omega). \tag{4.3}$$

Proof. Writing $\omega = D^{\text{MSO}}[f]$ and with reference to the diagram of (3.8), we have

$$\begin{aligned}
\phi^*\mathsf{p}_J^{\text{MSO}}(\omega) &= \phi^* f_!(\mathsf{p}_J(-\mathcal{V}_f^{\text{st}})) \\
&\overset{(3.9)}{=} \mu_!(\beta^*(\mathsf{p}_J(-\mathcal{V}_f^{\text{st}}))) \\
&= \mu_!(\mathsf{p}_J(-\beta^*\mathcal{V}_f^{\text{st}}) \\
&= \mu_!(\mathsf{p}_J(-\mathcal{V}_\mu^{\text{st}})) \\
&= \mathsf{p}_J^{\text{MSO}}(\omega^\mu)
\end{aligned}$$

with $\omega^\mu = D^{\text{MSO}}[\mu]$. The fourth equality is the stable equivalence (2.3) of $\mathcal{V}_\mu^{\text{st}}$ and $\beta^*(\mathcal{V}_f^{\text{st}})$. The claim, then, setting $\omega^f := \omega$, is that

$$\omega^\mu = \phi^*\omega^f. \tag{4.4}$$

But, from Sect.1, ω^f is the Pontryagin-Thom classifying map

$$X_+ \xrightarrow{f_+} M^{\mathcal{V}_f^{\text{st}}} \longrightarrow \text{MSO}_{-q},$$

which pulls-back by ϕ to

$$Y_+ \xrightarrow{\phi_+} X_+ \xrightarrow{f_+} M^{\mathcal{V}_f^{\text{st}}} \longrightarrow \text{MSO}_{-q},$$

while ω^μ is the Pontryagin-Thom classifying map

$$Y_+ \xrightarrow{\mu_+} M^{\mathcal{V}_\mu^{\text{st}}} \longrightarrow \text{MSO}_{-q},$$

and these fit together in the diagram

$$\begin{tikzcd}
Y_+ \arrow[rr, "\phi_+"] \arrow[d, "\mu_+"] & & X_+ \arrow[d, "f_+"] \\
(M \times_{\text{X}} Y)^{\mathcal{V}_\mu^{\text{st}}} \arrow[rr, "\beta_+"] \arrow[dr] & & M^{\mathcal{V}_f^{\text{st}}} \arrow[dl] \\
& \text{MSO}_{-q} &
\end{tikzcd} .$$

The top square we know to commute from (3.8), while the lower triangle commutes up to homotopy because $\mathcal{V}_\mu^{\text{st}} = \beta^*(\mathcal{V}_f^{\text{st}})$ in $\text{K}(M)$. Hence (4.4) holds in $[Y_+, \text{MSO}_{-q}]$, i.e. in $\text{MSO}^{-q}(Y)$. ∎

The characteristic classes (4.1) are well-defined invariants of the vertically oriented bordism class $[f, M, \alpha]$:

Theorem 4.3. *If* $(f, M, \alpha) \sim (f', M', \alpha')$ *(are bordant in* $\text{MSO}_*(X)$*) then*

$$f_!(\mathsf{p}_J(-\mathcal{V}_f^{\text{st}})) = f'_!(\mathsf{p}_J(-\mathcal{V}_{f'}^{\text{st}})) \tag{4.5}$$

in $\text{H}^*(X, \mathbb{Q})$ *for each* $J \in \mathbb{N}^\infty$. *Hence* $\mathsf{p}_J(D^{\text{MSO}}([f])) := f_!(\mathsf{p}_J(-\mathcal{V}_f^{\text{st}}))$ *gives a well-defined homomorphism of abelian groups*

$$\mathsf{p}_J^{\text{MSO}} : \text{MSO}^*(X) \otimes \mathbb{Q} \to \text{H}^*(X, \mathbb{Q}).$$

Proof. In singular cohomology the Poincaré duality construction (3.6) of the Umkehr map extends to vertically-oriented f. Precisely, Poincaré duality holds for a closed non-orientable manifold N in singular cohomology as a cap product isomorphism

$$\mathrm{H}^r(N,\mathbb{Z}) \xrightarrow{D} \mathrm{H}_{\dim N-r}(N,\mathbb{Z}^{\mathrm{sw}_1})$$

with $\mathrm{H}_k(N,\mathbb{Z}^{\mathrm{sw}_1})$ the homology of the twisted singular chain complex $S_k(\mathcal{O}_N)\otimes_{\mathbb{Z}[\mathbb{Z}_2]}\mathbb{Z}^-$ of the orientation double cover $\mathcal{O}_N$, where $\mathbb{Z}^-$ is $\mathbb{Z}$ as the right $\mathbb{Z}[\mathbb{Z}_2]$-module in which the generator of $\mathbb{Z}_2$ acts by -1, reducing to $\mathrm{H}_k(N,\mathbb{Z})$ if N is orientable; see Ch.4 of [Ranicki]. The point here is that $f : M \to X$ being vertically oriented means a specific isomorphism $\alpha_{\mathrm{M,X}} : \mathcal{O}_M \to f^*\mathcal{O}_X$ and hence a canonical bundle map $\mathcal{O}_M \to \mathcal{O}_X$ and hence a map of twisted chain complexes defining a push-forward

$$f_*^{\mathrm{sw}_1} : \mathrm{H}_k(M,\mathbb{Z}^{\mathrm{sw}_1}) \to \mathrm{H}_k(X,\mathbb{Z}^{\mathrm{sw}_1}),$$

and hence the extension of (3.6) as

$$f_! : \mathrm{H}^r(M,\mathbb{Z}) \xrightarrow{D} \mathrm{H}_{\dim M-r}(M,\mathbb{Z}^{\mathrm{sw}_1}) \xrightarrow{f_*^{\mathrm{sw}_1}} \mathrm{H}_{\dim M-r}(X,\mathbb{Z}^{\mathrm{sw}_1}) \xrightarrow{D^{-1}} \mathrm{H}^{r-q}(X,\mathbb{Z}). \tag{4.6}$$

This extends to a vertically oriented map $\sigma : W \to X$ on a manifold W with boundary $M := \partial W \neq \emptyset$ (defining a bordism) in an analogous manner as

$$\sigma_! : \mathrm{H}^r(W,M,\mathbb{Z}) \xrightarrow{D} \mathrm{H}_{\dim M-r}(W,\mathbb{Z}^{\mathrm{sw}_1}) \xrightarrow{\sigma_*^{\mathrm{sw}_1}} \mathrm{H}_{\dim M-r}(X,\mathbb{Z}^{\mathrm{sw}_1}) \xrightarrow{D^{-1}} \mathrm{H}^{r-q}(X,\mathbb{Z}), \tag{4.7}$$

where D, here, is Leftschetz-Poincaré duality.

For $(f,M,\alpha),(f',M',\alpha')$ representing elements of $\mathrm{MSO}_n(X)$, the difference element is represented by $(f\sqcup f', M\sqcup M', \alpha\sqcup(-\alpha'))$ with, using Lemma 4.1, vertical Pontryagin class

$$(f\sqcup f')_!(\mathsf{p}_J(-\mathcal{V}^{\mathrm{st}}_{(f\sqcup f',\alpha\sqcup(-\alpha'))})) = f_!(\mathsf{p}_J(-\mathcal{V}^{\mathrm{st}}_{(f,\alpha)})) - f'_!(\mathsf{p}_J(-\mathcal{V}^{\mathrm{st}}_{(f',\alpha')}))$$

(4.5) is thus equivalent to $(f\sqcup f')_!(\mathsf{p}_J(-\mathcal{V}^{\mathrm{st}}_{(f\sqcup f',\alpha\sqcup(-\alpha'))}))=0$ if $(f,M,\alpha)\sim(f',M',\alpha')$. So it is enough to show

$$f_!(\mathsf{p}_J(-\mathcal{V}^{\mathrm{st}}_f)) = 0 \quad \text{for} \quad (f,M,\alpha)\sim 0 \ \text{ in } \ \mathrm{MSO}_n(X), \tag{4.8}$$

that is, for $f : M \to X$ the restriction to $\partial W = M$ of vertically oriented $\sigma : W \to X$, so

$$\sigma\circ i = f \tag{4.9}$$

where $i : M \to W$ is the inclusion map, and check $f_!(\mathsf{p}_J(-\mathcal{V}^{\mathrm{st}}_f))$ is invariant with respect to fibrewise diffeomorphisms of $M \to N$ preserving the vertical orientation.

To see (4.8), there is the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^k(W,\mathbb{Z}) \arrow[r, "i^*"] \arrow[d] & \mathrm{H}^k(M,\mathbb{Z}) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^{k+1}(W,M,\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}^{k+1}(W,\mathbb{Z}) \arrow[d] \\
\mathrm{H}_{m+1-k}(W,M,\mathbb{Z}^{\mathrm{sw}_1}) \arrow[r] & \mathrm{H}_{m-k}(M,\mathbb{Z}^{\mathrm{sw}_1}) \arrow[r, "i_*"] & \mathrm{H}_{m-k}(W,\mathbb{Z}^{\mathrm{sw}_1}) \arrow[r] & \mathrm{H}_{m-k}(W,M,\mathbb{Z}^{\mathrm{sw}_1})
\end{tikzcd} \tag{4.10}$$

in which the vertical maps are the Poincaré duality maps and the horizontal sequences are exact. This is Prop 9., Ch. 8, of [Dold] modified to Poincaré duality for vertically oriented maps. Combining with (4.6) and (4.7) one obtains

$$\begin{tikzcd}
& \mathrm{H}_{m-k}(W, \mathbb{Z}^{\mathrm{sw}_1}) \arrow[dd, bend left] & & \mathrm{H}^{k+1}(W, M, \mathbb{Z}) \arrow[ll] \arrow[dd, dashed, bend left, "\sigma_!"] \\
\mathrm{H}_{m-k}(M, \mathbb{Z}^{\mathrm{sw}_1}) \arrow[ur, "i_*"] \arrow[dr, "f_*"'] & & \mathrm{H}^{k}(M, \mathbb{Z}) \arrow[ll] \arrow[ur, "\delta"] \arrow[dr, dashed, "f_!"] & \\
& \mathrm{H}_{m-k}(X, \mathbb{Z}^{\mathrm{sw}_1}) \arrow[rr] & & \mathrm{H}^{k-q}(X, \mathbb{Z})
\end{tikzcd} \tag{4.11}$$

(where $\mathrm{H}_{m-k}(W, \mathbb{Z}^{\mathrm{sw}_1}) \to \mathrm{H}_{m-k}(X, \mathbb{Z}^{\mathrm{sw}_1})$ is σ_*) in which the solid arrows form a commutative array. That commutativity implies that the right-hand triangle also commutes

$$f_! = \sigma_! \circ \delta. \tag{4.12}$$

Consider

$$\mathsf{p}_J(-\mathcal{V}_f^{\mathrm{st}}) \in \mathrm{H}^{4|J|}(W, \mathbb{Z}).$$

The boundary $M = \partial W$ has a collar neighbourhood $\mathcal{U}_M := \partial W \times [0,1) \subset W$ and so the restriction of T_W to ∂W has the form $i^* T_W \cong T_M + \underline{1}$. Hence

$$i^* \mathcal{V}_\sigma^{\mathrm{st}} = i^*(\sigma^* T_X - T_W) = i^* \sigma^* T_X - T_M - \underline{1} = f^* T_X - T_M - \underline{1} = \mathcal{V}_f^{\mathrm{st}} - \underline{1}. \tag{4.13}$$

Hence, as since p_J is stable, $\mathsf{p}_J(-\mathcal{V}_f^{\mathrm{st}}) = \mathsf{p}_J(-\mathcal{V}_f^{\mathrm{st}} + \underline{1}) = i^* \mathsf{p}_J(-\mathcal{V}_\sigma^{\mathrm{st}})$. Thus,

$$f_!(\mathsf{p}_J(-\mathcal{V}_f^{\mathrm{st}})) = f_! i^* \mathsf{p}_J(-\mathcal{V}_\sigma^{\mathrm{st}}) \overset{(4.9)}{=} \sigma_! \circ \delta(i^* \mathsf{p}_J(-\mathcal{V}_\sigma^{\mathrm{st}})) = \sigma_!(\delta \circ i^*)(\mathsf{p}_J(-\mathcal{V}_\sigma^{\mathrm{st}})) = 0$$

by the exactness of (4.10).

To see that $f_!(\mathsf{p}_J(-\mathcal{V}_f^{\mathrm{st}})) = g_!(\mathsf{p}_J(-\mathcal{V}_g^{\mathrm{st}}))$ for $f : M \to X$ and $g : N \to X$ vertically oriented maps with $f = g \circ \psi$ and $\psi : M \to N$ a vertical-orientation preserving diffeomorphism, one in this case has $T_M \cong \psi^* T_N$, and hence $\mathcal{V}_f^{\mathrm{st}} = \psi^* \mathcal{V}_g^{\mathrm{st}}$, and hence

$$\begin{aligned}
f_!(\mathsf{p}_J(-\mathcal{V}_f^{\mathrm{st}})) = g_! \psi_! \mathsf{p}_J(-\psi^* \mathcal{V}_g^{\mathrm{st}}) &= g_!(\psi_!(\psi^* \mathsf{p}_J(-\mathcal{V}_g^{\mathrm{st}}))) \\
&= g_!(\mathsf{p}_J(-\mathcal{V}_g^{\mathrm{st}}) \psi_!(1)) \\
&= g_!(\mathsf{p}_J(-\mathcal{V}_g^{\mathrm{st}}))
\end{aligned}$$

since for a diffeomorphism $\psi_!(1) = \pm 1$ with the positive sign taken if preserving the vertical orientation.

This continues to hold even for ψ a homeomorphism, preserving vertical orientation. For, in this case Novikov has shown that the rational Pontryagin classes are topological invariants — one has $\psi^* \mathsf{p}_j(N) = \mathsf{p}_j(M)$ over $\mathbb{Q}$. But $\mathsf{p}(N) = \mathsf{p}(-\mathcal{V}_g^{\mathrm{st}}) g^* \mathsf{p}(X)$, with p the total Pontryagin class, and a similar formula for $p(M)$. Equating using the Whitney sum formulae, one iteratively infers $\phi^* \mathsf{p}_j(-\mathcal{V}_g^{\mathrm{st}}) = \mathsf{p}_j(-\mathcal{V}_f^{\mathrm{st}})$ for all j, and the result follows as before. $\square$

This gives one direction of Theorem 1.

Corresponding statements are immediate for the vertical Chern and Stiefel-Whitney classes on, respectively, $\mathrm{MU}^*(X)$ and $\mathrm{MO}^*(X)$: one has

$$\mathsf{c}_J^{\mathrm{MU}} : \mathrm{MU}^*(X) \to \mathrm{H}^r(X, \mathbb{Z})$$

defined by
$$\mathsf{c}_J^{\mathrm{MU}}(D^{\mathrm{MU}}[f]) = f_!(\mathsf{c}_J(-\mathcal{V}_f^{\mathrm{st}})),$$
where $\mathsf{c}_J(-\mathcal{V}_f^{\mathrm{st}}) := \mathsf{c}_J(\zeta) = \mathsf{c}_{j_1}(\zeta)\cdots\mathsf{c}_{j_r}(\zeta)$ is the J^{th} Chern class of a stably equivalent complex virtual bundle $\zeta \in \mathrm{K}(M)$ defining the stable vertical complex structure on $-\mathcal{V}_f^{\mathrm{st}}$ — this is independent of the choice of ζ. Since a stable complex structure on $\mathcal{V}_f^{\mathrm{st}}$ defines an orientation on it, the above proof also implies
$$\mu = 0 \text{ in } \mathrm{MU}^*(X) \ \Rightarrow \mathsf{c}_J^{\mathrm{MU}}(\mu) = 0 \ \ \forall J \subset \mathbb{N}^\infty.$$
Likewise, vertical Stiefel-Whitney classes
$$\mathsf{sw}_J^{\mathrm{MO}} : \mathrm{MO}^*(X) \to \mathrm{H}^r(X, \mathbb{Z}_2)$$
are defined by
$$\mathsf{sw}_J^{\mathrm{MO}}(D^{\mathrm{MO}}[f]) = f_!(\mathsf{sw}_J(-\mathcal{V}_f^{\mathrm{st}})),$$
which collectively vanish when f is an unoriented boundary map.

In the latter case matters are simplified because any manifold is $\mathrm{H}\mathbb{Z}_2$-oriented, so the proof of Theorem 4.3 goes through unchanged with $\mathbb{Z}_2$ replacing $\mathbb{Z}$ and the twisted coefficients $\mathbb{Z}^{\mathrm{sw}_1}$.

Indeed, for any generalised cohomology theory $\mathbb{E}$ which is multiplicative with Poincaré duality and for which there is a stable $\mathbb{E}$-characteristic class $\beta : \mathrm{K}(N) \to \mathbb{E}(N)$, defining $\beta_J(D^{\mathbb{E}}[f]) := f_!(\beta_J(-\mathcal{V}_f^{\mathrm{st}}))$, if (4.10) commutes, then $\mu = 0$ in $\mathbb{E}^*(X) \Rightarrow \beta_J(\mu) = 0$ for all J, as in the proof of Theorem 4.3.

Comment 4.4. A more generic proof (avoiding twisted coefficients) uses the construction (3.3) of $f_!$, requiring only that f be vertically oriented; for this (4.11) is replaced by a diagram

$$\begin{tikzcd}
E^k(M) \arrow[rr, bend left, "\delta"] \arrow[d, "\phi^M"] \arrow[ddr, "f_!"] & & E^{k+1}(W,M) \arrow[d, "\phi^W"] \arrow[ddl, "\sigma_!"'] \\
E^{k-q}(M^{\mathcal{V}_f^{st}}) \arrow[dr, "P.T."'] & & E^{k-q}(W^{\mathcal{V}_\sigma^{st}}) \arrow[ll, bend right] \arrow[dl, "P.T."] \\
& E^{k-q}(X) &
\end{tikzcd}$$

where the unmarked arrow is induced by the map $M^{\mathcal{V}_f^{st}} \to W^{\mathcal{V}_\sigma^{st}}$ defined via (4.13). The rest of the proof proceeds unchanged when commutativity holds.

5. The converse: $\mathrm{P}_J^{\mathrm{MSO}}(\mu) = 0 \ \forall J \Rightarrow \mu = 0$ in $\mathrm{MSO}^*(X)$

For orientable X the result is inferred from classical results of Conner and Floyd, as follows. In this case $[f] \in \mathrm{MSO}_*(X)$ is represented by $f : M \to X$ with M $\mathrm{H}\mathbb{Z}$-orientable. Thus, one has generating fundamental classes $[M] \in \mathrm{H}_n(M, \mathbb{Z})$ and $[X] \in \mathrm{H}_n(X, \mathbb{Z})$.

A rational homology class $\beta \in \mathrm{H}_*(X, \mathbb{Q})$ is uniquely determined by the Kronecker pairing map

$$\langle \cdot\,, \beta\rangle : \mathrm{H}^*(X, \mathbb{Q}) \to \mathbb{Q}.$$

Define $q_J^{\text{MSO}}([f]) \in \mathrm{H}_*(X, \mathbb{Q})$ by setting – for arbitrary $\omega \in \mathrm{H}^*(X, \mathbb{Q})$ since M is orientable –

$$\langle \omega, q_J^{\text{MSO}}([f])\rangle \;=\; \langle f^*\omega \cup \mathsf{p}_J(-\mathcal{V}_f^{\text{st}}), [M]\rangle.$$

Using (3.5), the right-hand side is

$$\langle \omega \cup f_!(\mathsf{p}_J(-\mathcal{V}_f^{\text{st}})), [X]\rangle \;=\; \langle \omega \cup \mathsf{p}_J^{\text{MSO}}(D^{\text{MSO}}[f])), [X]\rangle \;=\; \langle \omega, D(\mathsf{p}_J^{\text{MSO}}(D^{\text{MSO}}[f]))\rangle$$

where the second equality uses the property $\langle \omega\cup\nu, [X]\rangle \;=\; \langle \omega, D\nu\rangle$ with D Poincaré duality on $\mathrm{H}^*(X, \mathbb{Q})$, and hence:

Lemma 5.1. $q_J^{\text{MSO}}([f]) \in \mathrm{H}_*(X, \mathbb{Q})$ *is Poincaré dual to* $\mathsf{p}_J^{\text{MSO}}(D^{\text{MSO}}[f]) \in \mathrm{H}^*(X, \mathbb{Q})$.

The vanishing of each of the classes $\mathsf{p}_J^{\text{MSO}}(D^{\text{MSO}}[f])$ therefore implies the same for the homology classes $q_J^{\text{MSO}}([f])$ and hence $\langle f^*\omega \cup \mathsf{p}_J(-\mathcal{V}_f^{\text{st}}), [M]\rangle = 0$ for each $\omega \in \mathrm{H}^*(X, \mathbb{Q})$ and $J \subset \mathbb{N}^\infty$ (specifically, for $4|J| + |\omega| = \dim M$), in this case. Taking

$$\omega = c \cup p_{J'}(X)$$

we infer that

$$\langle f^*c \cup \mathsf{p}_{J'}(f^*T_X) \cup \mathsf{p}_J(-\mathcal{V}_f^{\text{st}}), [M]\rangle = 0$$

for each $c \in \mathrm{H}^*(X, \mathbb{Q})$ and $J, J' \subset \mathbb{N}^\infty$. But $\mathsf{p}_I(M) = \sum_{J\sqcup J'=I} \mathsf{p}_{J'}(f^*T_X) \cup \mathsf{p}_J(-\mathcal{V}_f^{\text{st}})$ and hence

$$\langle f^*c \cup \mathsf{p}_I(M), [M]\rangle = 0 \tag{5.1}$$

for each $c \in \mathrm{H}^*(X, \mathbb{Q})$ and $I \subset \mathbb{N}^\infty$. The rational numbers on the left-hand side of (5.1) are the Conner-Floyd characteristic numbers for bordism classes of maps of oriented manifolds, whose collective vanishing was shown in [Conner, Floyd1] to imply (in fact, to be equivalent to) $[f] = 0$:

Thus, Theorem 1 holds when X is $\mathrm{H}\mathbb{Z}$-orientable.

Replacing p_J by sw_J and $\mathbb{Q}$ by $\mathbb{Z}_2$, the same argument proves the unoriented bordism version of Theorem 1 (the $\Rightarrow$ direction having been shown here in §3):

$$\alpha = \alpha' \text{ in } \mathrm{MO}^*(X) \quad\Longleftrightarrow\quad \mathsf{sw}_J^{\text{MO}}(\alpha) = \mathsf{sw}_J^{\text{MO}}(\alpha') \;\text{ in } \mathrm{H}^*(X, \mathbb{Z}_2) \;\; \forall\; J \subset \mathbb{N}^\infty. \tag{5.2}$$

$\mathrm{MO}^*(X)$ coincides with the unoriented bordism theory of [Conner, Floyd1] and (5.2) is equivalent to their result. A different proof of $\Leftarrow$ is given here below, and for $\eta\mathrm{O}_\mathrm{X}$ can alternatively be shown using a stable fibrewise Thom map.

However, our purpose is to prove Theorem 1 for $\mathrm{MSO}^*(X) \otimes \mathbb{Q}$, for which X need not be orientable, and also to prove the corresponding vertical Chern class characterisation of complex bordism cohomology $\mathrm{MU}^*(X)$ — that for $\mu \in \mathrm{MU}^*(X)$

$$\mathsf{c}_J^{\text{MU}}(\mu) = 0 \;\; \forall J \subset \mathbb{N}^\infty \;\;\Rightarrow \mu = 0 \text{ in } \mathrm{MU}^*(X), \tag{5.3}$$

the converse having been shown in §3. We proceed via a vertical Riemann-Roch theorem on $\mathrm{MU}^*(X)$. This adapts ideas of [Dyer] and [Buhštaber].

Let $\mathbb{h}^* = \{\mathbb{h}^k\}$ and $\mathbb{k}^* = \{\mathbb{k}^k\}$ be multiplicative cohomology theories. Let $\tau : \mathbb{h}^* \to \mathbb{k}^*$ be a natural transformation such that for each space N, $\tau : \mathbb{h}^*(N) \to \mathbb{k}^*(N)$ is a multiplicative (ring) homomorphism, and such that if $\alpha \in \mathbb{h}^1(S^1, \mathrm{pt})$ and $\beta \in \mathbb{k}^1(S^1, \mathrm{pt})$ are suspensions of the units in $\mathbb{h}^0(S^1, \mathrm{pt})$ and $\mathbb{k}^0(S^1, \mathrm{pt})$, then $\tau(\alpha) = \beta$. τ is then said to be a multiplicative transformation [Dyer].

Let ξ be a stable vector bundle over a manifold M, defining a class in the corresponding Kring (M) on M of stable bundles with a specified G-structure. A multiplicative transformation $\tau : \mathbb{h}^* \to \mathbb{k}^*$ may be used to associate to ξ a generalised Todd class

$$T_\tau(\xi) \in \mathbb{k}^*(M). \tag{5.4}$$

For this, suppose ξ is $\mathbb{h}$-oriented and $\mathbb{k}$-oriented, so one has Thom isomorphisms

$$\phi_{\mathbb{h}} = \phi_{\mathbb{h},\xi} : \mathbb{h}^r(M) \to \tilde{\mathbb{h}}^{r+\mathrm{rk}(\xi)}(M^\xi)$$

and

$$\phi_{\mathbb{k}} = \phi_{\mathbb{k},\xi} : \mathbb{k}^r(M) \to \tilde{\mathbb{k}}^{r+\mathrm{rk}(\xi)}(M^\xi).$$

Consider the composition

$$\mathbb{h}^*(M) \xrightarrow{\phi_{\mathbb{h}}} \tilde{\mathbb{h}}^*(M^\xi) \xrightarrow{\tau} \tilde{\mathbb{k}}^*(M^\xi) \xrightarrow{\phi_{\mathbb{k}}^{-1}} \mathbb{k}^*(M)$$

and assume that for each $n \in \mathbb{N}$

$$\tau(\phi_{\mathbb{h},\underline{n}}) = \phi_{\mathbb{k},\underline{n}}. \tag{5.5}$$

Then, define (5.4) by

$$T_\tau(\xi) = \phi_{\mathbb{k},\xi}^{-1}\,\tau\,\phi_{\mathbb{h},\xi}(1) = \phi_{\mathbb{k},\xi}^{-1}\,\tau(u_{\mathbb{h},\xi}(\xi))$$

with $u_{\mathbb{h}}(\xi) = \phi_{\mathbb{h},\xi}(1) \in \tilde{\mathbb{h}}^*(M^\xi)$ the Thom class. For $m \in \mathbb{h}^*(M)$ define

$$T_\tau(\xi)(m) = \phi_{\mathbb{k},\xi}^{-1}\,\tau\,\phi_{\mathbb{h},\xi}(m) \in \mathbb{k}^*(M).$$

Lemma 5.2. *The class $\xi \mapsto T_\tau(\xi)$ is stable, i.e. T_τ pushes-down to a group homomorphism $T_\tau : \mathrm{K}(M) \to \mathbb{k}^*(M)$ (from the appropriate* K*-theory).*

Proof. The Thom class has the functoriality property

$$\phi_{\mathbb{h}}(\xi + \eta) = p_1^*\phi_{\mathbb{h}}(\xi) \cup p_2^*\phi_{\mathbb{h}}(\eta) \tag{5.6}$$

where p_1, p_2 are the respective projection maps onto ξ and η, and similarly for $\phi_{\mathbb{k}}$. Hence, since τ is multiplicative,

$$T_\tau(\xi + \eta) = T_\tau(\xi)T_\tau(\eta). \tag{5.7}$$

From (5.5) (5.6) and (5.7) and that $T_\tau(\underline{n}) = 1_{\mathbb{h}}$, we have $T_\tau(\xi + \underline{n}) = T_\tau(\xi)$ and the result follows. □

Thus we have a canonical map $\mathrm{MSO}_*(X) \to \mathbb{k}^*(M)$ given by $[f] \mapsto T_\tau(\mathcal{V}_f^{\mathrm{st}})$, or, more concretely, relative to an embedding $e_l : M \to \mathbb{R}^l$ with oriented normal bundle $\mathcal{V}_{e_l \times f}$, by $[f] \mapsto T_\tau(\mathcal{V}_{e_l \times f})$, and similarly for $MU_*(X)$ with a (stable) complex structure class on $\mathcal{V}_{e_l \times f} + \underline{m}$ for m sufficiently large, the map is $[f] \mapsto T_\tau(\mathcal{V}_{e_l \times f} + \underline{m})$.

Lemma 5.3. [Dyer] *One has for a virtual bundle ξ*

$$T_\tau(\xi)(m) = T_\tau(\xi) \cup \tau(m).$$

Proof. It is enough, in view of Lemma 5.2, to prove this for a vector bundle ξ. Let $\pi : \xi \to M$ be the bundle projection map. Then, with $\phi_{\Bbbk} = \phi_{\Bbbk,\xi}, \phi_{\mathbb{h}} = \phi_{\mathbb{h},\xi}$,

$$\begin{aligned}
\phi_{\Bbbk}^{-1}\tau\phi_{\mathbb{h}}(m) &= \phi_{\Bbbk}^{-1}\tau(\phi_{\mathbb{h}}(1)\cup\pi^*m) \\
&= \phi_{\Bbbk}^{-1}(\tau(\phi_{\mathbb{h}}(1))\cup\tau(\pi^*m)) \\
&= \phi_{\Bbbk}^{-1}(\tau(\phi_{\mathbb{h}}(1))\cup\pi^*\tau(m)) \\
&= \pi_!(\tau(\phi_{\mathbb{h}}(1))\cup\pi^*\tau(m)) \\
&= \pi_!\left(\tau(\phi_{\mathbb{h}}(1))\right)\cup\tau(m) \\
&= \phi_{\Bbbk}^{-1}\tau(\phi_{\mathbb{h}}(1))\cup\tau(m)
\end{aligned}$$

□

The following is a tweak of Dyer's Riemann-Roch theorem for oriented maps in generalised cohomologies [Dyer] to the case of vertically oriented maps:

Theorem 5.4. *Let $f : M \to X$ be a continuous map of manifolds which is both $\mathbb{h}$ and $\Bbbk$ vertically oriented, so that the Umkehr maps $f_!^{\mathbb{h}}$ and $f_!^{\Bbbk}$ are defined and the vertical Todd class $T_\tau(\mathcal{V}_f^{\mathrm{st}})$ is defined. Then for $m \in \mathbb{h}^*(M)$ one has in $\Bbbk^*(M)$*

$$\tau(f_!^{\mathbb{h}}(m)) = f_!^{\Bbbk}(T_\tau(\mathcal{V}_f^{\mathrm{st}})\cup\tau(m)). \tag{5.8}$$

Proof. Since $\mathcal{V}_f^{\mathrm{st}}$ is $\mathbb{h}$-oriented and $\Bbbk$-oriented we have Thom isomorphisms

$$\phi_{\mathbb{h},\mathcal{V}_f^{\mathrm{st}}} : \mathbb{h}^r(M) \to \tilde{\mathbb{h}}^{r-q}(M^{\mathcal{V}_f^{\mathrm{st}}}), \quad \text{and} \quad \phi_{\Bbbk,\mathcal{V}_f^{\mathrm{st}}} : \Bbbk^r(M) \to \tilde{\Bbbk}^{r-q}(M^{\mathcal{V}_f^{\mathrm{st}}}).$$

Combining with the Pontryagin-Thom map $\tilde{f} : X_+ \to M^{\mathcal{V}_f^{\mathrm{st}}}$, inducing

$$\tilde{f}_{\mathbb{h}}^* : \tilde{\mathbb{h}}^*(M^{\mathcal{V}_f^{\mathrm{st}}}) \to \mathbb{h}^*(X_+) \quad \text{and} \quad \tilde{f}_{\Bbbk}^* : \tilde{\Bbbk}^*(M^{\mathcal{V}_f^{\mathrm{st}}}) \to \Bbbk^*(X_+),$$

gives the Umkehr/Gysin maps

$$f_!^{\mathbb{h}} = \tilde{f}_{\mathbb{h}}^*\circ\phi_{\mathbb{h}} : \mathbb{h}^r(M) \to \mathbb{h}^{r-q}(X) \quad \text{and} \quad f_!^{\Bbbk} = \tilde{f}_{\Bbbk}^*\circ\phi_{\Bbbk} : \Bbbk^r(M) \to \Bbbk^{r-q}(X), \tag{5.9}$$

and we have

$$\begin{aligned}
\tau(f_!^{\mathbb{h}}(m)) &= \tau\tilde{f}_{\mathbb{h}}^*\phi_{\mathbb{h}}(m) \\
&= \tilde{f}_{\Bbbk}^*\tau\phi_{\mathbb{h}}(m) \\
&= \tilde{f}_{\Bbbk}^*\phi_{\Bbbk}\left(\phi_{\Bbbk}^{-1}\tau\phi_{\mathbb{h}}(m)\right) \\
&= f_!^{\Bbbk}\left(\phi_{\Bbbk}^{-1}\tau\phi_{\mathbb{h}}(m)\right) \\
&= f_!^{\Bbbk}\left(T_\tau(\mathcal{V}_f^{\mathrm{st}})(m)\right)
\end{aligned}$$

and so using Lemma 5.3 and stability of T_τ we reach (5.8). □

In the case, making a stronger assumption, that M and X are $\mathbb{h}$, $\Bbbk$ orientable, one also has the (then) equivalent formulation of (5.8) of [Dyer]. Namely, M and X orientability gives Thom isomorphisms $\phi_{\mathbb{h},-T_M} : \mathbb{h}^r(M) \to \tilde{\mathbb{h}}^{r-q}(M^{-T_M})$ and $\phi_{\Bbbk,-T_M}$, and likewise for the stable normal bundle $-T_X$, yielding the construction (3.4) of the Umkehr maps $f_!^{\mathbb{h}}$ and $f_!^{\Bbbk}$ and generalised Todd classes of the stable normal bundles. Dyer's Riemann-Roch formula is then

$$\tau(f_!^{\mathbb{h}}(m))\cup T_\tau(-T_X) = f_!^{\Bbbk}(T_\tau(-T_M)(m)) \tag{5.10}$$

which in view of (3.5) and (5.7) is the same as (5.8). On the other hand, (5.8) holds for vertically oriented f when X is not orientable and (5.10) is not applicable.

The total Novikov operation [Adams, Quillen]

$$\mathbb{S} = \sum_J \mathbb{S}_J : \mathrm{MU}^*(X) \to \mathrm{MU}^*(X) \tag{5.11}$$

summed over multi-indices J, where $\mathbb{S}_J : \mathrm{MU}^k(X) \to \mathrm{MU}^{k+2|J|}(X)$, has the properties that $\mathbb{S}_0$ is the identity, that $f^*\mathbb{S}_J = \mathbb{S}_J f^*$, and $\mathbb{S}$ is multiplicative, and is of some tangential interest insofar as for a complex vector bundle $\xi \to M$

$$T_s(\xi) = \sum_J \mathsf{cf}_J^{\mathrm{MU}}(\xi)$$

with $\mathsf{cf}_J^{\mathrm{MU}}(\xi) = \mathsf{cf}_{j_1}^{\mathrm{MU}}(\xi)\cdots \mathsf{cf}_{j_r}^{\mathrm{MU}}(\xi)$, where $\mathsf{cf}_j^{\mathrm{MU}}(\xi)$ is the j^{th} Conner-Floyd Chern class of ξ [Adams, Kochman]. So (5.8) says:

Proposition 5.5. *For a vertical stably complex map $f : M \to X$ one has in* $\mathrm{MU}^*(X)$

$$\mathbb{S}(f_!^{\mathrm{MU}}(\mathsf{cf}_J^{\mathrm{MU}}(\xi))) = f_!^{\mathrm{MU}}(\mathsf{cf}_J^{\mathrm{MU}}(\mathcal{V}_f^{\mathrm{st}}) \cup \mathbb{S}(\mathsf{cf}_J^{\mathrm{MU}}(\xi))).$$

Here, any complex orientable cohomology theory has Conner-Floyd Chern classes associated to each complex bundle and these behave naturally with respect to Whitney sum and pull-back. The classes $\mathsf{cf}_k^{\mathbb{h}} \in \mathbb{h}^{2k}(\mathrm{BU})$ of the universal bundle provide a canonical set of generators for the polynomial ring $\mathbb{h}^*(\mathrm{BU}) = \mathbb{h}^*[\mathsf{cf}_1^{\mathbb{h}}, \mathsf{cf}_2^{\mathbb{h}}, \ldots]$ [Conner, Floyd2] , [Adams], [Kochman]. This mirrors the classical Chern classes – to which they reduce when $\mathbb{h} = \mathrm{H}\mathbb{Z}^*$ – however, the ring structure for general $\mathbb{h}$ is more subtle, depending on a formal group law.

The Novikov operation (5.11) is a homolog in MU^* of the Steenrod operation on $\mathrm{H}\mathbb{Z}_2$

$$Sq = \sum_{k=0} Sq_k : \mathrm{H}^*(N, \mathbb{Z}_2) \to \mathrm{H}^*(N, \mathbb{Z}_2),$$

where $Sq_k : \mathrm{H}^m(N, \mathbb{Z}_2) \to \mathrm{H}^{m+k}(N, \mathbb{Z}_2)$ is the k^{th} Steenrod square, for which one has the classical Stiefel-Whitney class formula

$$T_{Sq}(\zeta) = \sum_k \mathsf{sw}_k(\zeta) =: \mathsf{sw}(\zeta),$$

and (5.8) says for a real bundle $\zeta \to M$

$$Sq(f_!^{\mathrm{H}\mathbb{Z}_2}(\mathsf{sw}(\zeta))) = f_!^{\mathrm{H}\mathbb{Z}_2}(\mathsf{sw}(\mathcal{V}_f^{\mathrm{st}}) \cup Sq(\mathsf{sw}(\zeta))) \quad \text{in } \mathrm{H}^*(X, \mathbb{Z}_2),$$

similarly to [Atiyah, Hirzebruch] and [Dyer].

The multiplicative transformation we have specific need of here is the rational Chern-Dold character which defines a canonical map from a multiplicative cohomology $\mathbb{h}^*$

$$\mathsf{ch}^{\mathbb{h}} : \mathbb{h}^*(N) \otimes \mathbb{Q} \longrightarrow \mathrm{H}^*(N, \Lambda_{\mathbb{h}}^* \otimes \mathbb{Q}) := \sum_{l \geq 0} \mathrm{H}^l(N, \Lambda_{\mathbb{h}}^{l-1} \otimes \mathbb{Q})$$

to singular cohomology with coefficients in the rational cobordism ring $\Lambda_{\mathbb{h}}^* \otimes \mathbb{Q}$. On finite complexes $\mathsf{ch}^{\mathbb{h}}$ is a ring isomorphism. For, [Dyer] Cor.4 proves that the rationalisation of any multiplicative (co)homology theory is singular cohomology, that there is an isomorphism $\mathbb{h}^*(N) \otimes \mathbb{Q} \cong \mathrm{H}^*(N, \Lambda_{\mathbb{h}}^* \otimes \mathbb{Q})$ and by [Dyer] Th.2 there is a unique such isomorphism which on $N = \mathrm{pt}$ coincides with the identity map

$\Lambda^*_{\mathbb{h}} \otimes \mathbb{Q} \to \Lambda^*_{\mathbb{h}} \otimes \mathbb{Q}$ – this is $\mathsf{ch}^{\mathbb{h}}$.

We now apply Theorem 5.4 to the Chern-Dold character on vertical bordism cohomology.

First, note how it applies to the classical case $\mathbb{h} = \mathrm{K}, \mathbb{k} = \mathrm{H}^*$ for which the Chern-Dold character reduces to the usual Chern character $\mathsf{ch}^{\mathrm{U}} : \mathrm{K}(N) \otimes \mathbb{Q} \longrightarrow \mathrm{H}^*(N, \mathbb{Q})$, and for ξ a stable complex bundle

$$T(-\xi) = \phi^{-1}_{\mathrm{H},-\xi} \mathsf{ch}^{\mathrm{U}} \phi_{\mathrm{K},-\xi}(1) = \mathsf{Td}(\xi) \tag{5.12}$$

with Td the classical Todd class, which augments for $m = E \in \mathrm{K}(M)$ to

$$\phi^{-1}_{\mathrm{H},-\xi} \mathsf{ch}^{\mathrm{U}} \phi_{\mathrm{K},-\xi}(E) = \mathsf{Td}(\xi)\, \mathsf{ch}^{\mathrm{U}}(E),$$

(see for example §12 of [Lawson, Michelson]). By (5.8), the Riemann-Roch theorem therefore extends to vertically oriented $f : M \to X$ as

$$\mathsf{ch}^{\mathrm{U}}(f^{\mathrm{K}}_!(E)) = f^{\mathrm{H}}_!(\mathsf{Td}(-\mathcal{V}^{\mathrm{st}}_f)\, \mathsf{ch}^{\mathrm{U}}(E)). \tag{5.13}$$

K-orientability means $\mathcal{V}^{\mathrm{st}}_f$ is spin-c and in this case $\mathsf{Td}(\mathcal{V}^{\mathrm{st}}_f) = e^{c_1(\mathcal{V}^{\mathrm{st}}_f)/2} \widehat{\mathsf{A}}(\mathcal{V}^{\mathrm{st}}_f)$. If M and X are assumed to be K-orientable almost complex manifolds (and H-orientable), then $\mathsf{Td}(-\mathcal{V}^{\mathrm{st}}_f)) = \mathsf{Td}(M) f^*(\mathsf{Td}(X))^{-1}$ and matters reduce to a result of [Atiyah, Hirzebruch] (§3 eq.$(i)'$) and [Dyer], and (5.13) takes its customary form

$$\mathsf{ch}^{\mathrm{U}}(f^{\mathrm{K}}_!(E))\mathsf{Td}(X) = f^{\mathrm{H}}_!(\mathsf{ch}^{\mathrm{U}}(E)\mathsf{Td}(M)).$$

Similarly, with $\mathbb{h} = \mathrm{KO}, \mathbb{k} = \mathrm{H}^*$ and $\tau = \mathsf{ch}^{\mathrm{SO}}$ the Pontryagin character one has

$$\mathsf{ch}^{\mathrm{SO}}(f^{\mathrm{KO}}_!(E)) = f^{\mathrm{H}}_!(\widehat{\mathsf{A}}(-\mathcal{V}^{\mathrm{st}}_f))\, \mathsf{ch}^{\mathrm{SO}}(E))$$

reverting when M and X are spin manifolds (KO-orientable) to

$$\mathsf{ch}^{\mathrm{SO}}(f^{\mathrm{KO}}_!(E))\widehat{\mathsf{A}}(X) = f^{\mathrm{H}}_!(\widehat{\mathsf{A}}(M)\, \mathsf{ch}^{\mathrm{SO}}(E)).$$

Now contemplate the case

$$\mathbb{h} = \mathrm{MU}^* \otimes \mathbb{Q}, \qquad \mathbb{k} = \mathrm{H}^*(\quad, \Lambda^*_{\mathrm{MU}} \otimes \mathbb{Q}),$$

and the Chern-Dold character

$$\tau := \mathsf{ch}^{\mathrm{MU}} : \mathrm{MU}^*(N) \otimes \mathbb{Q} \longrightarrow \mathrm{H}^*(N, \Lambda^*_{\mathrm{MU}} \otimes \mathbb{Q}).$$

A rank n complex vector bundle ξ is both MU and H oriented. Let

$$u_\xi = u_{\mathrm{MU},\xi} \in \mathrm{MU}^{2n}(M) \tag{5.14}$$

be the Thom class corresponding to the homotopy class of maps $M^\xi \to \mathrm{MU}_n$ defined by the classifying class $M \to \mathrm{BU}_n$ of ξ. Let $\phi_\xi = \phi_{\mathrm{MU},\xi} : \mathrm{MU}^r(M) \to \widetilde{\mathrm{MU}}^{r+2n}(M^\xi)$ be the resulting Thom isomorphism, $\phi_{\mathrm{MU},\xi}(1) = u_\xi$. Let $\phi_{\mathrm{H},\xi}(1)$ be the Thom class in $\mathrm{H}^*(M, \Lambda^*_{\mathrm{MU}} \otimes \mathbb{Q})$ determined by the Thom class $\mu_{\mathbb{Z}}(\phi_{\mathrm{MU},\xi}(1)) \in \mathrm{H}^*(M, \mathbb{Z})$, where $\mu_{\mathbb{Z}} : \mathrm{MU}^*(M) \to \mathrm{H}^*(M, \mathbb{Z})$ is the Thom/Steenrod homomorphism in cohomology, and let $\phi_{\mathrm{H},\xi}$ be the resulting Thom isomorphism. Then, mirroring (5.12), there is the Buchstaber-Todd class [Buhštaber]

$$\mathsf{Td}^{\mathrm{MU}}(\xi) := T(-\xi) = \phi_{\mathrm{H},-\xi}\, \mathsf{ch}^{\mathrm{MU}}\, \phi_{\mathrm{MU},-\xi}(1) \ \in \mathrm{H}^*(M, \Lambda^*_{\mathrm{MU}} \otimes \mathbb{Q}).$$

Applying (5.8) we infer that

$$\mathsf{ch}^{\mathrm{MU}}(\underbrace{f_!^{\mathrm{MU}}(1)}_{\in \mathrm{MU}^{-q}(X)}) = f_!^{\mathrm{H}}(\mathsf{Td}^{\mathrm{MU}}(\mathcal{V}_f^{\mathrm{st}})) = f_!^{\mathrm{H}}(\phi_{\mathrm{H},\mathcal{V}_f^{\mathrm{st}}}\,\mathsf{ch}^{\mathrm{MU}}\,\phi_{\mathrm{MU},\mathcal{V}_f^{\mathrm{st}}}(1)).$$

To evaluate these terms, we have from (5.9) that for $[f : M \to X] \in \mathrm{MU}_{\dim M}(X)$

$$f_!^{\mathrm{MU}}(1) = \tilde{f}^*_{\mathrm{MU}} \circ \phi_{\mathrm{MU},\mathcal{V}_f^{\mathrm{st}}}(1), \qquad 1 \in \mathrm{MU}^0(M).$$

By the definition of (5.14), the Thom class $\phi_{\mathrm{MU},\mathcal{V}_f^{\mathrm{st}}}(1) = u_{\mathcal{V}_f^{\mathrm{st}}}$ associated to $[f]$ is the element of MU cohomology specified by the classifying map $M^{\mathcal{V}_f^{\mathrm{st}}} \longrightarrow \mathrm{MU}_{-q}$, while $\tilde{f}^*_{\mathrm{MU}}$ is induced by the Pontryagin-Thom map $X_+ \to M^{\mathcal{V}_f^{\mathrm{st}}}$. Their composition is thus precisely the vertical Atiyah-Poincaré dual element $D^{\mathrm{MU}}[f] \in \mathrm{MU}^{-q}(X)$ computed in Sect.1. That is,

Proposition 5.6.

$$f_!^{\mathrm{MU}}(1) = D^{\mathrm{MU}}[f].$$

If one additionally assumes that M and X are individually stable complex manifolds then the Poincaré duality construction (3.6) of $f_!^{\mathrm{MU}}$ can be used to rederive

$$f_!^{\mathrm{MU}}(1) = D^{\mathrm{MU}} \circ f_*^{\mathrm{MU}} \circ (D^{\mathrm{MU}})^{-1}(1_{\mathrm{MU}^*}) = D^{\mathrm{MU}} \circ f_*^{\mathrm{MU}}([1]) = D^{\mathrm{MU}}([f]),$$

where $[1]$ is the bordism class of the identity map $1 : M \to M$.

By Theorem 5.8 we so far have:

Proposition 5.7.

$$ch^{\mathrm{MU}}(D^{\mathrm{MU}}[f]) = f_!^{\mathrm{H}}(Td^{\mathrm{MU}}(-\mathcal{V}_f^{\mathrm{st}})). \tag{5.15}$$

This extends Th. 1.5 of [Buhštaber] to vertically oriented $f : M \to X$.

By stability (Lemma 5.2) $\mathsf{Td}^{\mathrm{MSO}}(\mathcal{V}_f^{\mathrm{st}}) = \mathsf{Td}^{\mathrm{MU}}(\mathcal{V}_{e_l \times f} + \underline{m})$ for the normal bundle $\mathcal{V}_{e_l \times f}$ with $l >> 0$ and m such that $\mathcal{V} := \mathcal{V}_{e_l \times f} + \underline{m}$ has a complex structure defining the stable complex class of $\mathcal{V}_f^{\mathrm{st}}$. Buhštaber (loc. cit.) proves that for any complex vector bundle ξ

$$\mathsf{Td}^{\mathrm{MU}}(\xi) = \exp\left\{\sum_{i \geq 1} \frac{[N^{2i}]}{d_i}\mathsf{ch}_i^{\mathrm{U}}(\xi)\right\}$$

for unique up to bordism stably complex manifolds N^{2i}, and certain specific integers d_i. From this we infer an expansion

$$\mathsf{Td}^{\mathrm{MU}}(-\mathcal{V}_f^{\mathrm{st}}) = \sum_J b_J\,[M_J]\,\mathsf{c}_J(-\mathcal{V}_f^{\mathrm{st}}) \;\in \mathrm{H}^*(M, \Lambda^*_{\mathrm{MU}} \otimes \mathbb{Q}) \tag{5.16}$$

for some closed manifolds M_J and $b_J \in \mathbb{Q}$ – (5.16) can alternatively be deduced from the identity

$$\mathsf{ch}^{\mathrm{MU}}(\mathsf{cf}_1^{\mathrm{MU}}(\xi)) = \mathsf{ch}_1^{\mathrm{U}}(\xi) + \sum_{n \geq 1}[M^{2n}]\,\mathsf{ch}_{n+1}^{\mathrm{U}}(\xi) \tag{5.17}$$

of [Buhštaber] (Cor. 2.4). From (5.15)

$$\mathsf{ch}^{\mathrm{MU}}(D^{\mathrm{MU}}[f]) = \sum_J b_J\,[M_J]\, f_!^{\mathrm{H}}\left(\mathsf{c}_J(-\mathcal{V}_f^{\mathrm{st}})\right) = \sum_J b_J\,[M_J]\,\mathsf{c}_J^{\mathrm{MU}}(D^{\mathrm{MU}}[f]). \tag{5.18}$$

Hence

$$\mathsf{c}_J^{\mathrm{MU}}(D^{\mathrm{MU}}[f]) = 0 \ \ \forall\ J \subset \mathbb{N}^\infty \implies \mathsf{ch}^{\mathrm{MU}}(D^{\mathrm{MU}}[f]) = 0$$

and hence

$$\mu := D^{\mathrm{MU}}[f] = 0 \text{ in } \mathrm{MU}^*(X) \otimes \mathbb{Q}$$

since $\mathsf{ch}^{\mathrm{MU}}$ is a $\mathbb{Q}$ ring isomorphism, or equivalently $[f] = 0$ in $\mathrm{MU}_*(X)\otimes\mathbb{Q}$. Since any cohomology class in $\mathrm{MU}^*(X)$ can be so written, this completes the proof of (5.3).

Let us sketch the proof for $\mathrm{MO}^*(X)$ and $\mathrm{MSO}^*(X)\otimes\mathbb{Q}$. Note, first, that MO is a real oriented cohomology theory, every real vector bundle $\xi \to X$ is MO-oriented with Thom class $u_\xi \in \tilde{\mathrm{MO}}^n(X^\xi)$. Equivalently, there is a $w_{\mathrm{MO}} \in \tilde{\mathrm{MO}}^1(\mathbb{R}P^\infty)$ mapping to 1 in $\tilde{\mathrm{MO}}^1(\mathbb{R}P^\infty) \to \tilde{\mathrm{MO}}^1(\mathbb{R}P^1) \to \tilde{\mathrm{MO}}^1(S^1) \to \tilde{\mathrm{MO}}^0(*)$, defining the first universal Conner-Floyd Stiefel-Whitney class $\mathsf{cf}_1^{\mathrm{MO}}(L^\infty) := w_{\mathrm{MO}} \in \mathrm{MO}^1(\mathbb{R}P^\infty)$ of the tautological line bundle $L^\infty \to \mathbb{R}P^\infty$. The AHSS shows that $\mathrm{MO}^*(\mathbb{R}P^\infty) = \mathrm{MO}^*[w]$. Pull-back by the classifying map and the Grothendieck construction yield a full spread of Conner-Floyd Stiefel-Whitney classes $\mathsf{cf}_k^{\mathrm{MO}}(\xi) \in \mathrm{MO}^k(X)$. Naturality holds for Whitney sum and pull-back, while the ring structure depends on a logarithm of the formal group law

$$l_{\mathrm{MO}}(T) = \sum_{i\geq 0} s_i T^i$$

with $s_i \in \mathrm{MO}^*(pt)$ such that $\mathsf{cf}_1^{\mathrm{MO}}(l_{\mathrm{MO}}(\zeta_1\otimes\zeta_2)) := \mathsf{cf}_1^{\mathrm{MO}}(\zeta_1) + \mathsf{cf}_1^{\mathrm{MO}}(\zeta_2)$ for real line bundles ζ_1, ζ_2 [Quillen]. The s_i are readily determined.

Theorem 5.8.

$$\alpha = \alpha' \text{ in } \mathrm{MO}^*(X) \iff \mathsf{sw}_J^{\mathrm{MO}}(\alpha) = \mathsf{sw}_J^{\mathrm{MO}}(\alpha') \ \text{ in } \mathrm{H}^*(X,\mathbb{Z}_2) \ \ \forall\ J\subset\mathbb{N}^\infty.$$

Proof. From [Quillen] there is a canonical isomorphism

$$\mathrm{MO}^*(X) \longrightarrow \mathrm{H}^*(X,\mathbb{Z}_2)\otimes\Lambda^*_{\mathrm{MO}} \tag{5.19}$$

which reduces to the identity map on $X = pt$ and so by Dold's theorem [Dold] must coincide with the Chern-Dold character $\mathsf{ch}^{\mathrm{MO}}$. Concretely, the inverse to (5.19) is identified by Quillen to be the Λ^*_{MO} extension of the map $\mathrm{H}^*(X,\mathbb{Z}_2) \to \mathrm{MO}^*(X)$ which sends an element of $\mathrm{H}^*(X,\mathbb{Z}_2)$, identified with a real line bundle L, to $l_{\mathrm{MO}}(\mathsf{sw}_1^{\mathrm{MO}}(L))$. Let $\mathsf{Td}^{\mathrm{MO}}(-\mathcal{V}_f^{\mathrm{st}}) \in \mathrm{H}^*(X,\mathbb{Z}_2)\otimes\Lambda^*_{\mathrm{MO}}$ be the associated MO Todd class. The identity $f_!^{\mathrm{MO}}(1) = D^{\mathrm{MO}}[f]$ holds just as in (5.6) (via the forgetful functor) and hence from (5.8) we have that $\mathsf{ch}^{\mathrm{MO}}(D^{\mathrm{MO}}[f]) = f_!^{\mathrm{H}}(\mathsf{Td}^{\mathrm{MO}}(-\mathcal{V}_f^{\mathrm{st}}))$. Entirely similar proofs to those of [Buhštaber] (Lem 1.2 and latter part of Thm 1.4) yield

$$\mathsf{Td}^{\mathrm{MO}}(-\mathcal{V}_f^{\mathrm{st}}) = \sum_J [M_J]\,\mathsf{sw}_J(-\mathcal{V}_f^{\mathrm{st}})$$

for some $[M_J] \in \Lambda^*_{\mathrm{MO}}$ (we do not need any precision on the coefficients). Hence $\Leftarrow$ follows on applying the Umkehr map. The direction $\Rightarrow$ has been shown in §3. $\square$

For oriented bordism, with $\mathbb{h} = \mathrm{MSO}^* \otimes \mathbb{Q}, \mathbb{k} = \mathrm{H}^*(\ \ , \Lambda^*_{\text{MSO}} \otimes \mathbb{Q}), \tau = \mathsf{ch}^{\text{MSO}}$, and $\phi_{\text{MSO},\xi}$ the Thom isomorphism defined by the homotopy class of the classifying map $M^\xi \to \mathrm{MSO}_n$, an application of Theorem 5.4 gives in a similar way as for MU^*

$$\mathsf{ch}^{\text{MSO}}(D^{\text{MSO}}[f]) = f_!^{\text{H}}(\mathsf{Td}^{\text{MSO}}(-\mathcal{V}_f^{\text{st}}))$$

where $\mathsf{Td}^{\text{MSO}}(\xi) := \phi_{\text{H},-\xi}\, \mathsf{ch}^{\text{MSO}}\, \phi_{\text{MSO},-\xi}(1) \ \in \mathrm{H}^*(M, \Lambda^*_{\text{MSO}} \otimes \mathbb{Q})$.

Since $\mathrm{K}(M)\otimes\mathbb{Q} \to \mathrm{H}^*(M, \Lambda^*_{\text{MSO}}\otimes\mathbb{Q}),\ \xi \mapsto \mathsf{Td}^{\text{MSO}}(\xi)$, is a characteristic class – insofar as it satisfies Whitney sum additivity (5.7) and pulls-back functorially (because Thom isomorphisms and the Chern-Dold character do so) – then, by uniqueness of the Pontryagin classes, $\mathsf{Td}^{\text{MSO}}(-\mathcal{V}_f^{\text{st}})$ is of the form

$$\mathsf{Td}^{\text{MSO}}(-\mathcal{V}_f^{\text{st}}) = \sum_J b_J\,[M_J]\,\mathsf{p}_J(-\mathcal{V}_f^{\text{st}}) \tag{5.20}$$

for some $[M_J] \in \Lambda^*_{\text{MSO}}$ and $b_J \in \mathbb{Q}$. It thus follows as before that

$$\mathsf{p}_J^{\text{MSO}}(D^{\text{MSO}}[f]) = 0 \ \ \forall\ J \subset \mathbb{N}^\infty \implies D^{\text{MU}}[f] = 0 \text{ in } \mathrm{MSO}^*(X)\otimes\mathbb{Q},$$

completing the proof of Theorem 1.

In fact, the form of the expansion (5.20) can be seen directly using $\mathsf{Td}^{\text{MSO}}(\mathcal{V}_f^{\text{st}}) = \mathsf{Td}^{\text{MSO}}(\mathcal{V}_{e_l\times f})$ where for $l >> 0$ the normal bundle $\mathcal{V}_{e_l\times f}$ is a stable representative and setting $\mathcal{V} := \mathcal{V}_{e_l\times f}\otimes\mathbb{C}$. Let $u_{\text{MSO}} \in \tilde{\mathrm{MSO}}^{l-q}(X)$ be its Thom class. Define the top MSO Conner-Floyd Pontryagin class of $\mathcal{V}$ as the Euler class

$$\mathsf{cf}^{\text{MSO}}_{l-q}(\mathcal{V}) := e^{\text{MSO}}(\mathcal{V}) := i^*_{\text{MSO}} u_{\text{MSO}}.$$

Using this it is known how to define a total MSO Conner-Floyd Pontryagin class $\mathsf{cf}^{\text{MSO}}(\mathcal{V}) = \sum_{j\geq 0}\mathsf{cf}^{\text{MSO}}_j(\mathcal{V})$ in a natural way, via Grothendieck's construction, such that $\mathsf{cf}^{\text{MSO}}(\mathcal{V}\oplus\mathcal{V}') = \mathsf{cf}^{\text{MSO}}(\mathcal{V})\mathsf{p}^{\text{MSO}}(\mathcal{V}')$ and $\mathsf{cf}^{\text{MSO}}(\mathcal{V}\oplus\underline{n}) = \mathsf{cf}^{\text{MSO}}(\mathcal{V})$ over $\mathbb{Q}$, see [Conner, Floyd2], [Quillen]. We have

$$\begin{aligned}
i_!\mathsf{ch}^{\text{MSO}}(\mathsf{cf}^{\text{MSO}}_{l-q}(\mathcal{V}_f^{\text{st}})) = i_!\mathsf{ch}^{\text{MSO}}(\mathsf{p}^{\text{MSO}}_{l-q}(\mathcal{V})) &:= i_!\mathsf{ch}^{\text{MSO}}(i^*_{\text{MU}}u_{\mathcal{V}}) \\
&= i_!i^*_{\text{H}}\,\mathsf{ch}^{\text{MSO}}(u_{\mathcal{V}}) \\
&= \mathsf{ch}^{\text{MSO}}(u_{\mathcal{V}}) \cup u_{\mathcal{V}} \\
&= \phi_{\text{H}}\phi_{\text{H}}^{-1}(\mathsf{ch}^{\text{MSO}}(u_{\mathcal{V}})) \cup u_{\mathcal{V}} \\
&= i_!(\mathsf{Td}^{\text{MSO}}(\mathcal{V})) \cup u_{\mathcal{V}} \\
&= i_!(\mathsf{Td}^{\text{MSO}}(\mathcal{V}) \cup i^*u_{\mathcal{V}}) \\
&= i_!(\mathsf{Td}^{\text{MSO}}(\mathcal{V}) \cup \mathsf{p}_{l-q}(\mathcal{V})),
\end{aligned}$$

and so $\mathsf{ch}^{\text{MSO}}(\mathsf{cf}^{\text{MSO}}_{l-q}(\mathcal{V})) = \mathsf{Td}^{\text{MSO}}(\mathcal{V})\cup\mathsf{p}_{l-q}(\mathcal{V})$. The result now follows from an expansion of the desired form of $\mathsf{ch}^{\text{MSO}}(\mathsf{cf}_{l-q}(\mathcal{V}))$, which in turn holds as a consequence of the splitting principle and the expansion of $\mathsf{ch}^{\text{MSO}}(\mathsf{cf}^{\text{MSO}}_1(\mathcal{V}))$ in classical Pontryagin classes, as in the complex case from (5.17), a fact which is a consequence of the formal group law on MSO^* (or more easily on MSp^*) as in [Buhštaber].

6. Proof of Theorem 2.

An oriented cobordism genus is a ring homomorphism $\mathrm{MSO}^*(X)\otimes\mathbb{Q} \to \mathrm{H}^*(X,\mathbb{Q})$, and a complex genus is a ring homomorphism $\mathrm{MU}^*(X)\otimes\mathbb{Q}\to\mathrm{H}^*(X,\mathbb{Q})$. Just as in the classical case such objects are seen to be defined by multiplicative sequences. Recall, for the latter one considers the algebra $\mathcal{H}[z] = 1 + \sum_{j\leq 1} z^j\mathrm{H}^j(M,\mathbb{Q})$ of polynomials $b(z) = 1 + b_1 z + b_2 z^2 + \dots$ in a formal variable z with $b_j \in \mathrm{H}^j(M, \mathbb{Q})$, and an algebra endomorphism κ of $\mathcal{H}[z]$. If the coefficients $\kappa_j(b_1, \dots, b_j)$ of

$$\kappa(b(z)) = 1 + \kappa_1(b_1)z + \kappa_2(b_1, b_2)z^2 + \dots$$

are polynomials, homogeneous relative to b_j being assigned degree j, they are said to define a multiplicative sequence in view of the endomorphism property of κ

$$\kappa(a(z) \cdot b(z)) = \kappa(a(z)) \cdot \kappa(b(z)). \tag{6.1}$$

A multiplicative sequence determines and is determined by a power series $g \in \mathbb{C}[[z]]$ via the correspondence $\kappa(1+z) = g(z)$ and $1 + b_1 z + b_2 z^2 + \dots = \Pi_{i=1}^M (1 + \beta_i z)$ with $\beta_i \in \mathrm{H}^{k_i}(M, \mathbb{Q})$, from which (6.1) gives $\kappa(b(z)) = \Pi_{i=1}^M g(\beta_i z)$, identifying b_j with the i^{th} elementary symmetric function of $\beta_1, \dots, \beta_M$. On a closed $4m$-dimensional manifold Y this is applied to its total Pontryagin class $1 + p_1 + \dots + p_m$ to define the oriented genus $\Phi_\kappa(Y) = \int_Y \kappa(p) = \int_Y \kappa_m(p_1, \dots, p_m) \ \in \mathbb{Q}$. Similarly, multiplicative sequences in Chern classes lead to complex genera.

Vertical genera are constructed in the same way but using the Pontryagin class of the vertical stable normal bundle and integration over the fibre. Precisely:

Theorem 6.1. *For each rational multiplicative sequence* $\{\kappa_m\}$*, in the notation of* (4.1)*, the correspondence*

$$\begin{aligned} \omega \to \kappa([f^\omega]) &= f_!^\omega(\kappa(p(-\mathcal{V}_{f^\omega}^{\text{st}}))) \\ &= \sum_m \underbrace{f_!^\omega\left(\kappa_m\left(p_1(-\mathcal{V}_{f^\omega}^{\text{st}}), \dots, p_m(-\mathcal{V}_{f^\omega}^{\text{st}})\right)\right)}_{\in \mathrm{H}^{4m-q}(X,\mathbb{Q})}, \end{aligned} \tag{6.2}$$

defines a vertical genus $\mathrm{MSO}^*(X) \to \mathrm{H}^*(X, \mathbb{Q})$ *natural with respect to pull-back.*

Analogous statements hold for vertical complex genera in terms of Chern classes and vertical unoriented genera $\mathrm{MO}^*(X) \to \mathrm{H}^*(X, \mathbb{Z}_2)$ in terms of vertical Stiefel-Whitney classes.

Proof. By Theorem 4.3 the map (6.2) is a vertical cobordism invariant. From the commutativity of (2.9), the ring product of $\omega^f = D^{\text{MSO}}[f]$, $\omega^g = D^{\text{MSO}}[g] \in \mathrm{MSO}^*(X)$ is the element

$$\omega^{f \times_X g} = D^{\text{MSO}}[f \times_X g] \in \mathrm{MSO}^*(X).$$

From the identity $\mathcal{V}_{f \times_X g}^{\text{st}} = \mu^* \mathcal{V}_f^{\text{st}} + \nu^* \mathcal{V}_g^{\text{st}}$ of Proposition 2.2 for transverse maps $f : M \to X$ and $g : N \to X$ defining a commutative diagram

$$\begin{tikzcd} M \times_X N \arrow[r, "\mu"] \arrow[d, "\nu"] \arrow[dr, "f \times_X g"] & M \arrow[d, "f"] \\ N \arrow[r, "g"] & X, \end{tikzcd}$$

we have

$$\mathsf{p}(-\mathcal{V}_{f \times_X g}^{\text{st}}) = \mu^* \mathsf{p}(-\mathcal{V}_f^{\text{st}}) \cup \nu^* \mathsf{p}(-\mathcal{V}_g^{\text{st}}) \tag{6.3}$$

in $\mathrm{H}^*(M \times_X N)$ and hence

$$\kappa(\mathsf{p}(-\mathcal{V}^{\mathrm{st}}_{f\times_X g})) = \mu^*\kappa(\mathsf{p}(-\mathcal{V}^{\mathrm{st}}_f)) \cup \nu^*\kappa(\mathsf{p}(-\mathcal{V}^{\mathrm{st}}_g)).$$

From $f \times_X g = g \circ \nu$ we have for $a \in \mathrm{H}^*(N,\mathbb{Q}), b \in \mathrm{H}^*(M,\mathbb{Q})$

$$\begin{aligned}(f \times_X g)_!(\mu^*a \cup \nu^*b) &= g_! \circ \nu_!(\mu^*a \cup \nu^*b)\\ &= g_!(\nu_!\mu^*a \cup b)\\ &= g_!(g^*f_!a \cup b)\\ &= f_!a \cup g_!b\end{aligned}$$

using (3.9) for the third equality. Consequently,

$$\begin{aligned}(f \times_X g)_!(\kappa(\mathsf{p}(-\mathcal{V}^{\mathrm{st}}_{f\times_X g}))) &= (f \times_X g)_!(\mu^*\kappa(\mathsf{p}(-\mathcal{V}^{\mathrm{st}}_f)) \cup \nu^*\kappa(\mathsf{p}(-\mathcal{V}^{\mathrm{st}}_g)))\\ &= f_!(\kappa(-\mathsf{p}(\mathcal{V}^{\mathrm{st}}_f))) \cup g_!(\kappa(\mathsf{p}(-\mathcal{V}^{\mathrm{st}}_g))).\end{aligned}$$

(6.2) hence defines a ring homomorphism, and so a vertical genus.

That the genus (6.2) is natural for pull-back is the equality

$$g^*(f_!(\kappa(p(-\mathcal{V}^{\mathrm{st}}_f)))) = \mu_!(\kappa(p(-\mathcal{V}^{\mathrm{st}}_\mu))).$$

But by (3.9) we have $g^* \circ f_! = \mu_! \circ \nu^*$ from which the above identity follows on recalling $\mu^*\mathcal{V}^{\mathrm{st}}_f = \mathcal{V}^{\mathrm{st}}_\nu$ and the functoriality of the Pontryagin classes. $\square$

Example 6.2 (Vertical $\hat{A}$-genus on fibrations)**.** Consider fibre bundles $\pi : M \to X$ and $\pi' : M' \to X$, with product

$$\begin{tikzcd} M \times_X M' \arrow[r, "b'"] \arrow[d, "b"] \arrow[dr, "\pi\times_X\pi'"] & M' \arrow[d, "\pi'"] \\ M \arrow[r, "\pi"] & X. \end{tikzcd}$$

Let $p^\pi = 1 + p_1^\pi + \dots$ be the Pontryagin class of the stable normal bundle $-\mathcal{V}^{\mathrm{st}}_\pi$ of π. Since π is a submersion there is a (non canonical) vector bundle isomorphism

$$T_M \cong T_\pi + \pi^*T_X$$

where $T_\pi := \mathrm{Ker}\,(d\pi)$ is the tangent bundle along the fibres (the vertical tangent bundle). Thus

$$-\mathcal{V}^{\mathrm{st}}_\pi = T_\pi$$

is an honest vector bundle and a vertical orientation is an orientation on each fibre M_x in the usual sense, while $q = \dim M_x$. The J^{th} vertical Pontryagin class is

$$\mathsf{p}_J(\pi) := \pi_!(\mathsf{p}_J(T_\pi)) \in \mathrm{H}^*(X,\mathbb{Q}).$$

The $\hat{A}$-class on M of $-\mathcal{V}^{\mathrm{st}}_\pi = T_\pi$ is a sum

$$\hat{A}^\pi = 1 + \hat{A}_4^\pi + \hat{A}_8^\pi + \cdots \in \mathrm{H}^{4*}(M,\mathbb{Q})$$

with

$$\hat{A}_4^\pi = -\frac{p_1^\pi}{24}, \qquad \hat{A}_8^\pi = \frac{7(p_1^\pi) - 4p_2^\pi}{5760}, \qquad \dots$$

giving the vertical $\hat{A}$-genus in $\mathrm{H}^*(X,\mathbb{Q})$

$$\hat{A}(\pi) := \pi_!(\hat{A}^\pi) = \underbrace{\hat{A}_{4-q}(\pi)}_{\in \mathrm{H}^{4-q}(X,\mathbb{Q})} + \underbrace{\hat{A}_{8-q}(\pi)}_{\in \mathrm{H}^{8-q}(X,\mathbb{Q})} + \dots \quad ,$$

where

$$\hat{A}_{4-q}(\pi) = \pi_!\left(-\frac{p_1^\pi}{24}\right), \qquad \hat{A}_{8-q}(\pi) = \pi_!\left(\frac{7(p_1^\pi)^2 - 4p_2^\pi}{5760}\right), \qquad \dots \quad . \tag{6.4}$$

Thus

$$\hat{A}(\pi)\cdot\hat{A}(\pi) = \underbrace{\hat{A}_{4-q}(\pi)\hat{A}_{4-q'}(\pi')}_{\in \mathrm{H}^{8-(q+q')}(X,\mathbb{Q})} + \underbrace{\hat{A}_{4-q}(\pi)\hat{A}_{8-q'}(\pi') + \hat{A}_{8-q}(\pi')\hat{A}_{4-q}(\pi)}_{\in \mathrm{H}^{12-(q+q')}(X,\mathbb{Q})} + \dots$$

which by (1.10) is expected to coincide with

$$\hat{A}(\pi\times_X\pi') = \underbrace{\hat{A}_{8-(q+q')}(\pi\times_X\pi')}_{\in \mathrm{H}^{8-(q+q')}(X,\mathbb{Q})} + \underbrace{\hat{A}_{12-(q+q')}(\pi\times_X\pi')}_{\in \mathrm{H}^{12-(q+q')}(X,\mathbb{Q})} + \dots \quad .$$

To check the first of these equalities $\hat{A}_{8-(q+q')}(\pi\times_X\pi') = \hat{A}_{4-q}(\pi)\hat{A}_{4-q'}(\pi')$ note

$$\hat{A}_{4-q}(\pi)\hat{A}_{4-q'}(\pi') = \frac{1}{576}p_1(\pi)p_1(\pi') \tag{6.5}$$

from (6.4), and

$$\hat{A}_{8-(q+q')}(\pi\times_X\pi') = (\pi\times_X\pi')_!\left(\frac{7(p_1^{\pi\times_X\pi'})^2 - 4p_2^{\pi\times_X\pi'}}{5760}\right).$$

But

$$(p_1^{\pi\times\pi'})^2 = (b^*p_1^* + (b')^*p_1^{\pi'})^2 = b^*(p_1^\pi p_1^\pi) + (b')^*(p_1^{\pi'}p_1^{\pi'}) + 2b^*p_1^\pi(b')^*p_1^{\pi'},$$

and since $(\pi\times_X\pi')_! = (\pi\circ b)_! = \pi_!\circ b_! = \pi'_!\circ b'_!$ and $b_!b^* = 0$, then

$$(\pi\times_X\pi')_!(p_1^{\pi\times\pi'})^2 = 2\pi_!b_!(b^*p_1^\pi(b')^*p_1^{\pi'}) = 2p_1(\pi)p_1(\pi')$$

by the same steps as in the proof of Theorem 6.1. On the other hand, from (6.3)

$$p_2^{\pi\times_X\pi'} = b^*p_2^\pi + (b')^*p_2^{\pi'} + b^*p_1^\pi(b')^*p_1^{\pi'},$$

so

$$(\pi\times_X\pi')(p_2^{\pi\times_X\pi'}) = \pi_!b_!(b^*p_1^\pi(b')^*p_1^{\pi'}) = p_1(\pi)p_1(\pi').$$

Thus

$$\begin{aligned}(\pi\times_X\pi')_!\left(\frac{7(p_1^{\pi\times_X\pi'}) - 4p_2^{\pi\times_X\pi'}}{5760}\right) &= \frac{7\times 2p_1(\pi)p_1(\pi') - 4p_1(\pi)p_1(\pi')}{5760}\\ &= \frac{1}{576}p_1(\pi)p_1(\pi'),\end{aligned}$$

which is 6.5.

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